Math 4329

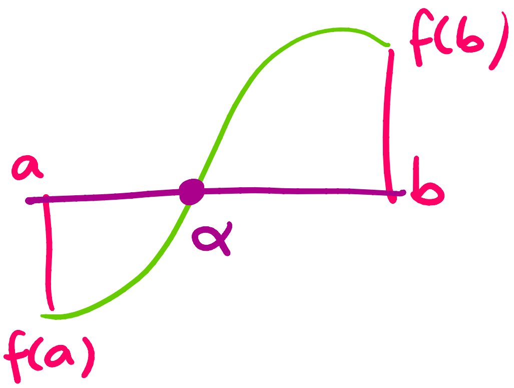
Tuesday, February 25, 2020 4:30 PM

chap 3 → Root finding techniques

Bisection Method

$\alpha \to$ Root of

$f(x)=0$.



Obtain $[0,1]$

c_1

c_2

$c_3 \dots$

$|\alpha - c_n| < \varepsilon$

$$|\alpha - c_n| \leq \frac{b-a}{2^n} < \varepsilon$$

↑ True valve

approximate vale

$[0,1]$ $f(x) = e^{-x} - x$, $\varepsilon = 10^{-6}$

$n \geq 19.9.. \approx 20$ $n \geq 20$

Bisection efficient or not?

Advantages of Bisection

① Reliable

② Only needs evaluation of $f(x)$ at a, b, c.

Disadvantage: slow.

method

Disadvantage: slow.

Measure the speed of convergence of a method by ORDER OF CONVERGENCE. (OOC)

$e_1 = \alpha - c_1$ → error at 1st iteration of method

$e_2 = \alpha - c_2$ → " " 2nd " "

$\vdots$

$e_n = \alpha - c_n$ → error at the nth iteration of method

for a given iteration method (here we only know Bisection)

the Order of convergence for the method is said to be p if

$$|e_{n+1}| \leq c\,|e_n|^p$$

$c > 0$ & if $p = 1$ then $c < 1$

→ error at n+1th iteration smaller than error at the previous iteration.

Bisection Method:

$$|\alpha - c_2| < \frac{b-a}{2^2}$$

$$|\alpha - c_1| < \frac{b-a}{2}$$

$$|\alpha - c_2| < \tfrac{1}{2}\,|\alpha - c_1|^1$$

$$|\alpha - c_2| < \frac{1}{2} |\alpha - c_1|^1$$

Here $p=1$ and the Order of convergence for Bisection is linear.

$$|e_{n+1}| \leq C |e_n|^{\textcircled{p}}$$

→ p speed/order of convergence.

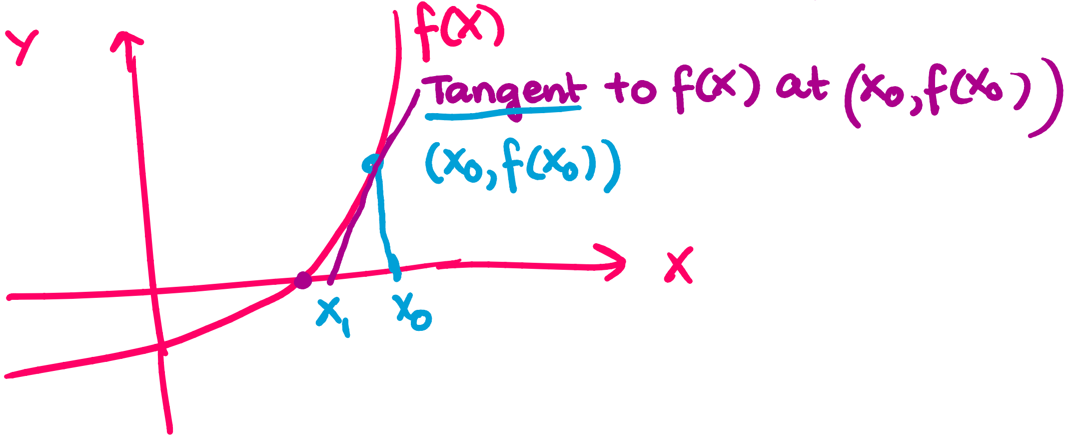
NEWTON'S METHOD :

- → What is Logic?
- → Why?
- → Applicability & limitation?
- → Order of convergence?? $|e_{n+1}| \leq c|e_n|^2$

Newton's method has Quadratic Convergence.

find α :

$f(x)=0$



Initial Guess

x_0

→ Newton formula

x_1 = Based on x_0 & $f(x)$.

$f(x_1)$ & check $|x_1 - x_0| < \varepsilon$ the stop else.

Construct x_2 based on x_1 & $f(x)$ & NEWTON'S FORMULA.

What is this formula? Based on x_0,

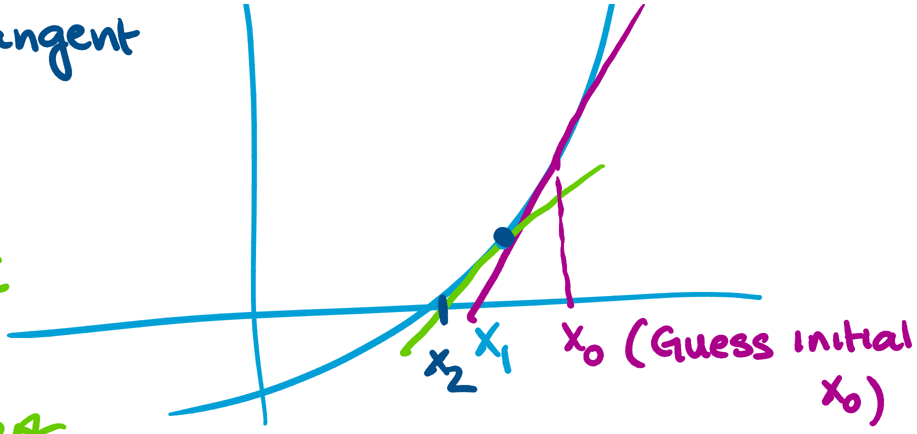
x-intercept of tangent

x-intercept of tangent to $\{x_0, f(x_0)\}$.

$\rightarrow x_1$

check $|f(x_1)| < \varepsilon$ and $|x_1 - x_0| < \varepsilon$ then stop & accept x_1 as the root. else continue to finding x-intercept of tangent at $(x_1, f(x_1))$

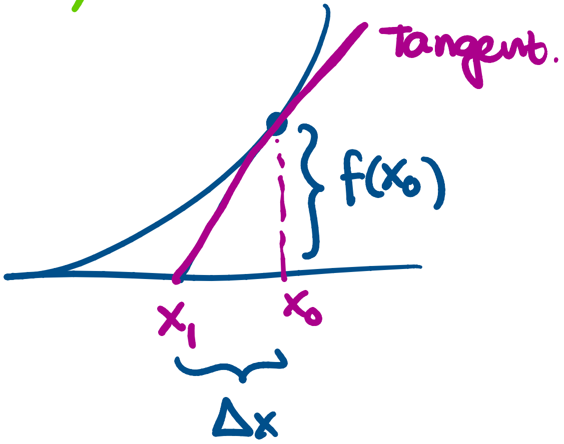
$\downarrow x_2$



Math. formula for Newton:

$$x_0 - x_1 = \Delta x$$

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{f(x_0)}{\Delta x}$$

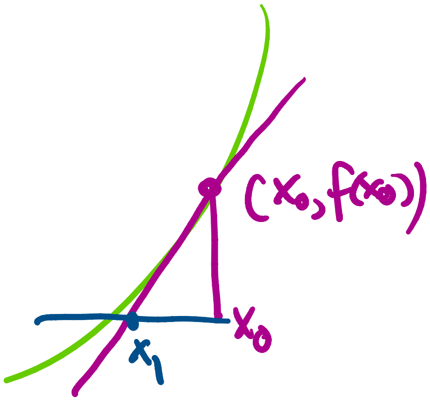


$$f'(x_0) = \frac{f(x_0)}{\Delta x}$$

$\Delta x \rightarrow$ increment from x_0

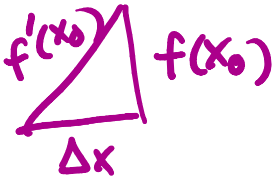
$$x_0 - x_1 = \Delta x$$

$$x_1 = x_0 - \Delta x$$



we know $\Delta x = f(x_0) / f'(x_0)$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$



$|f(x_1)| < \varepsilon$ & $|x_1 - x_0| < \varepsilon$ thenstop

else $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$|f(x_2)| < \varepsilon$ & $|x_2 - x_1| < \varepsilon$

Application of Newton's Method: By using the operations $+, -, \div, *$ approximate the square root of 3 iteratively given $\varepsilon = 10^{-6}$.

find $\sqrt{3}$. $\rightarrow$ Taylor poly (exam1)

can think of finding the root of

$x = \sqrt{3} \rightarrow x^2 = 3$

$x^2 - 3 = 0$

$x^2 - 3 = 0$.

$f(x) = x^2 - 3$

apply Newton's method to $f(x) = x^2 - 3$ with $\varepsilon = 10^{-6}$.

Guess x_0 construct $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

check if $|f(x_1)| < \varepsilon$ & $|x_1 - x_0| < \varepsilon$

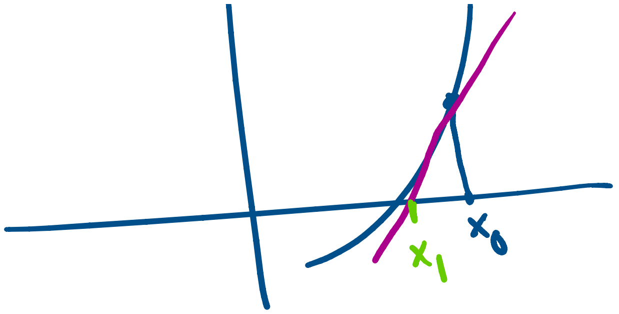
if true then stop & accept x_1 as root

else construct

$x_2 = x_1 - f(x_1) / f'(x_1)$

$\sqrt{3} \approxeq 1.7320508\,75 = \alpha$

$x - 1.7$



$x_0 = 1.7$

$f(x) = x^2 - 3$

$f'(x) = 2x$

$$x_1 = x_0 - f(x_0) / f'(x_0)$$

$$= 1.7 - \frac{((1.7)^2 - 3)}{(2*1.7)} = 1.73235$$

$$|f(x_1)| = |1.73235^2 - 3| = 0.001036 > \varepsilon$$

$$|\boxed{\alpha} - x_1| \xrightarrow{} |x_0 - x_1|$$

$$|\alpha - x_1| = |1.7320508 - 1.73235|$$

$$= 0.0002992 = 2.9 * 10^{-4}$$

$$|x_2 - x_1| \not\longleftarrow |\alpha - x_1|$$

$$|x_1 - x_0| \leftarrow |\alpha - x_0|$$

$\boxed{x_0 = 1.7}$ $\alpha = \underline{\underline{1.7320508}}\ldots$ $|\alpha - x_0| = 3.2 * 10^{-2}$

$$|\alpha - x_0| = \begin{array}{r} 1.7320508 \\ -\,1.7 \\ \hline 0.0320508 \end{array} \rightarrow \boxed{3.2 * 10^{-2}}$$

__Notice:__ $|\alpha - x_1| = 2.9 * 10^{-4}$

$$|\alpha - x_0| = 3.2 * 10^{-2}$$

$$|\alpha - x_1| \le C\,|\alpha - x_0|^2$$

$$|\alpha - x_1| \leq C\,|\alpha - x_0|^2$$

If we calculate x_2

$$|\alpha - x_2| \simeq 2.6 * 10^{-8}$$

Newton's method is faster than Bisection provided x_0 is chosen close to α.

Given x_0

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

What if $f'(x_0)$ DNE?

$f(x) = \log x$

$f'(x) = 1/x \quad x_0 = 0$

$$x_1 = x_0 - \frac{f(x_0)}{\boxed{f'(x_0)}} \rightarrow \text{nonzero.}$$

$x_0 = -1 \quad x_1 = 1 \quad x_2 = -1 \quad x_3 = 1 \ldots$

To overcome calculation of $f'(x_0)$

$$f'(x_0) \simeq \frac{f(x_0) - f(x_*)}{x_0 - x_*}$$

x_* is also an initial guess.