Math-4329

Thursday, February 6, 2020 4:37 PM

MATLAB session (see end of the notes for more)

Homework 01 problem 2 use $-0.5 \le x \le 0.5$.

Loss-of-Significance Error.

$$\frac{1-\cos(x)}{x^2}, \quad x = 10^{-6}, 10^{-7}, 10^{-8}$$

from Calculus $\displaystyle\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$ $\quad \dfrac{f(x)}{g(x)}$

$$\to \frac{1-\cos 0}{0^2}$$

$\to 0/0$ indeterminate form

In Calculus, we dealt with this by

L'Hopital Rule

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{-(-\sin x)}{2x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\cos x}{2} \to 1/2$$

$x \to 0 \qquad \frac{}{2}$

Use Taylor polynomial to avoid the loss-of-significance errors in evaluating $\frac{1-\cos x}{x^2}$ as $x \to 0$.

$f(x) = \cos x$ degree 2 poly. $p_2(x) = \cos 0 - \sin 0\, x - \cos 0 \frac{x^2}{2}$

$= 1 - x^2/2$

$$\frac{1-\cos x}{x^2} \underset{\text{approximated by}}{\approx} \frac{1-(1-x^2/2)}{x^2} = \frac{1-1+x^2/2}{x^2} = 1/2$$

Can we use degree 4 poly for cos x? Yes!

$$p_4(x) = \cos 0 - \cos 0 \frac{x^2}{2} + \sin 0 \frac{x^3}{3!} + \cos 0 \frac{x^4}{4!}$$

$$= 1 - x^2/2 + x^4/24$$

$$\cos x \approx p_4(x)$$

↳ approximated by

$$\frac{1-\cos x}{x^2} \approx \frac{1-(1-x^2/2+x^4/24)}{x^2}$$

$$= \frac{1-1+x^2/2-x^4/24}{x^2}$$

$$= \frac{1}{x^2}\left(x^2/2 - x^4/24\right)$$

$$= 1/2 - x^2/24$$

$x = 10^{-6}, 10^{-7}, 10^{-8}$

$\longrightarrow 1/2$ as $x \to 0$

$\approx 1/$

$\longrightarrow 1/2$ as $x \to 0$

$\approx 1/2$

Problem arises when we subtract 2 numbers v. close to each other

$\dfrac{e^x - e^{-x}}{2x} \longrightarrow$ cause a loss-of-sig error.

$e^x \xrightarrow{\text{deg 2 Taylor poly}} 1 + x + x^2/2$

$e^{-x} \xrightarrow{\text{deg 2 Taylor}} 1 - x + x^2/2$

$$\frac{e^x - e^{-x}}{2x} \overset{\text{Taylor poly}}{\approx} \frac{(1 + x + x^2/2) - (1 - x + x^2/2)}{2x}$$

$$= \frac{1 + x + x^2/2 - 1 + x - x^2/2}{2x}$$

$$= \frac{2x}{2x} = 1$$

Check using L'Hopital Rule:

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{2x} \quad \left(\frac{1-1}{0}\right) \quad \frac{0}{0}$$

Apply L'Hopital Rule

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{2x} \longrightarrow \lim_{x \to 0} \left(\frac{e^x - (e^{-x} * -1)}{2} \right)$$

$$\lim_{x \to 0} \frac{\quad}{2x} \qquad \lim_{x \to 0} \frac{\quad}{2}$$

$$\lim_{x \to 0} \frac{e^x + e^{-x}}{2} \to \frac{e^0 + e^0}{2} = 1$$

(More fun: exercise 6, Page 54).

Some details from MATLAB SESSION. For more, look at the Matlab Introduction Commands on the Course website

```
>> x=rand(1)

x =

   0.9575

>> n =x*100

n =

  95.7507

>> x=linspace(-0.5,0.5,1e+3);
>> plot(x, exp(cos(x)),'m*-')
>>
```