Thursday, January 23, 2020 4:31 PM

Last class: $f(x) = e^x \cos x$

Without calculating $P_3(x)$,

$|f(x) - P_3(x)| \leq ?$ $-\pi \leq x \leq \pi$.

$$f(x) - P_3(x) = \frac{(x-a)^4}{4!} f^{(4)}(c_x) \quad \text{(Taylor Remainder formula)}$$

$a = 0$ $c_x \to$ unknown number bet. x and 0.

$$f^{(4)}(c_x) = -4e^{c_x} \cos c_x$$

$$|f(x) - P_3(x)| = \left| \frac{(x-0)^4}{4!} (-4 e^{c_x} \cos c_x) \right| \quad -\pi \leq x \leq \pi$$

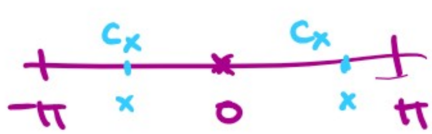
$$= \left| \frac{x^4}{\cancel{4!}} \right| \left| \cancel{4} e^{c_x} \cos c_x \right|$$

$3*2$

$-\pi \leq x \leq \pi$

$$= \frac{|x|^4}{6} \left| e^{c_x} \cos c_x \right| \leq \frac{\pi^4}{6} \left| e^{c_x} \cos c_x \right|$$

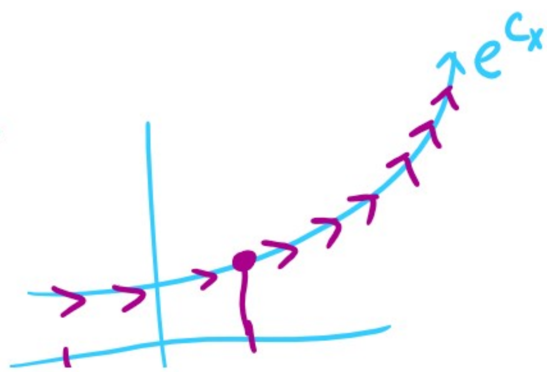
Pick c_x so that $|e^{c_x} \cos c_x| \leq$ Max value

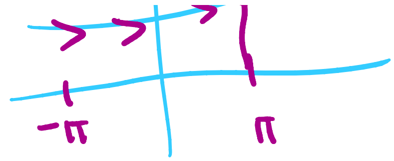


c_x also lines bet. $-\pi$ and π.

$|\cos \pi| \leq 1$

$e^{\pi} \geq e^{c_x}$

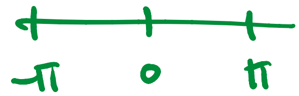




$$|e^{c_x}\cos c_x| \le |e^{\pi}\cos\pi|$$
$$= |e^{\pi} * -1| = e^{\pi}$$

$$|f(x)-P_3(x)| = \left|\frac{x^4}{6}\right| |e^{c_x}\cos c_x|$$

$-\pi \le x \le \pi$



$x=\pi$ $\quad$ $c_x=\pi$

$$\le \frac{\pi^4}{6} |e^{\pi}\cos\pi|$$

$$= \frac{\pi^4}{6} e^{\pi}$$

$16.23 * e^{\pi} \approx 375$

$f(x)=e^x\cos x$

$|f(x)-P_3(x)| \le ?$ $\quad$ $\boxed{0\le x\le \pi/2}$

$$|f(x)-P_3(x)| = \left|\frac{x^4}{6}(e^{c_x}\cos c_x)\right| \quad 0\le x\le \frac{\pi}{2}$$

$$= \left|\frac{x^4}{6}\right| * |e^{c_x}\cos c_x| \quad 0\le x\le \pi/2$$

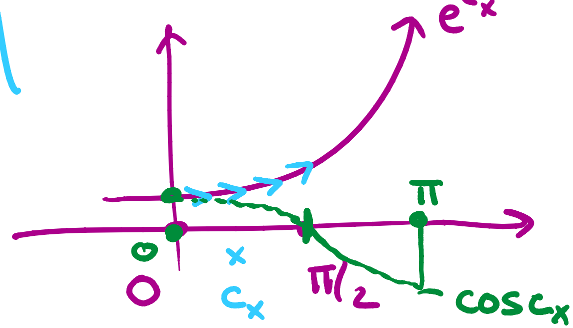
$0\le x\le \pi/2$

$|x|^4 \le ?$

$$\le \frac{(\pi/2)^4}{6} |e^{c_x}\cos c_x|$$

$c_x=\pi/2$

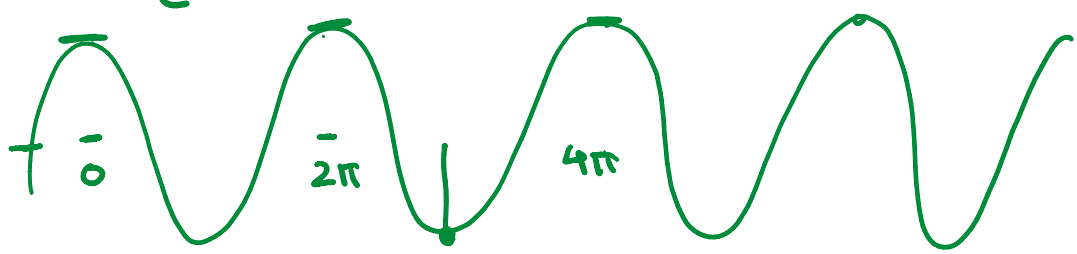
$e^{\pi/2}$



$0 \quad c_x \quad \pi/2 \quad \cos c_x$

<u>Note</u>: $|e^{c_x} \cos c_x| \leq ??$ $\quad 0 \leq c_x \leq \pi/2$

$$\underbrace{|e^{c_x}|}_{e^{\pi/2}} * \underbrace{|\cos c_x|}_{1} = e^{\pi/2}$$



without calculating $p_4(x)$, find an upper bound for $|f(x) - p_4(x)|$, $f(x) = e^{-x}$, $0 \leq x \leq 1$.

concluding previous ques:

$$|f(x) - p_3(x)| \leq \frac{(\pi/2)^4}{6} * e^{\pi/2} = \frac{\pi^4}{2^4 * 6} e^{\pi/2}$$

$$\frac{6.088}{6} * 4.810 \simeq 4.88$$

↳ Approximately

$$\frac{\overset{0.0001}{|\text{True value}} - \overset{0.0009}{\text{approx}|}}{|\text{True value}|}$$ Relative error

$f(x) = e^{-x} \qquad 0 \le x \le 1.$

$|f(x) - P_4(x)| \le \quad ?$

$$= \frac{(x-0)^{4+1}}{5!} f^{(5)}(c_x)$$

c_x lies bet. x & 0.

How to pick "a"? $\quad \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c_x)$

$$P_n(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \dots$$
$$f^{(n)}(a)(x-a)^n/n!$$

$f(x) = e^{-x} \qquad 0 \le x \le 1$

$|f(x) - P_4(x)| = \left|\frac{x^5}{5!} f^{(5)}(c_x)\right| \qquad 0 \le x \le 1$

$f(x) = e^{-x} = \frac{1}{e^x}$

$f'(x) = \frac{d}{dx}(e^{-x}) = e^{-x} * -1 = -e^{-x}$

$f''(x) = -(-e^{-x}) = (-1)^2 e^{-x}$

$\vdots$

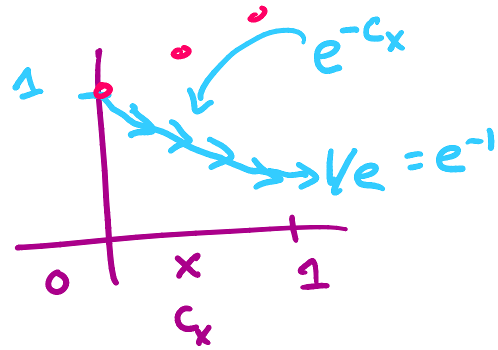
$f^{(5)}(x) = -e^{-x}$

$|f(x) - P_4(x)| = \frac{|x^5|}{5!} * |-e^{-c_x}| \qquad \underline{\underline{0 \le x \le 1}}$

$\le \frac{1}{5!} * \boxed{e^{-c_x}} \qquad 0 \le x \le 1$

$$\leq \frac{1}{5!} * \boxed{e^{-c_x}} \qquad 0 \leq x \leq 1$$

$$e^{-c_x} = \frac{1}{e^{c_x}}$$



$c_x = 0$

$$\rightarrow |f(x) - P_4(x)| \leq \frac{1}{5!} e^0 = \frac{1}{60} = 0.0166667$$

$$\approx 0.0167$$

Approximate $\sqrt{2}$ using $\underline{\underline{a}}$ Taylor poly. of deg 2.

$P_2(x)$ centered at a.

$\sqrt{x}$ $\qquad$ $\sqrt{x+1}$

$f(x) = \sqrt{x}$ and then calculate $p_2(x)$ &

$f(2) = \sqrt{2} \approx p_2(2)$

$$p_2(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2}$$

$f(x) = \sqrt{x}$ & $a = 0$

$f(0) = 0$

$$f'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = 1/2\, x^{1/2-1} = 1/2\, x^{-1/2}.$$

$$f'(x) = \frac{1}{2\sqrt{x}} \rightarrow f'(0) = \frac{1}{2*\sqrt{0}}$$ Doesnot exist!

$f'(x) = \frac{1}{2\sqrt{x}} \longrightarrow f'(0) = \frac{1}{2*\sqrt{0}}$ Doesnot exist!

a cannot be 0. $\boxed{\text{choose } a=1}$

$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2*1} = 1/2$

$f''(x) = \frac{d}{dx}\left(1/2\, x^{-1/2}\right) = -\frac{1}{4} x^{-1/2-1} = -\frac{1}{4} x^{-3/2}$

$f''(1) = -1/4 * 1 = -1/4$

$P_2(x) = f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2}$

$= 1 + 1/2(x-1) + (-1/4)\frac{(x-1)^2}{2}$

$\sqrt{2} = f(2) \approx P_2(2) = 1 + 1/2 - 1/4 \frac{(2-1)^2}{2}$

$= 1.5 - 1/8$

$= 1.375$ (approx. value)

1.4142 (True value)

How large should the Taylor polynomial $P_n(x)$ (centered at 0) be so that
$\underset{a=0}{}$

$|\underbrace{\sin x}_{f(x)} - P_n(x)| \le \underbrace{10^{-5}}_{\text{error tolerance}}$ $\quad -\pi/4 \le x \le \pi/4$.

$P_1(x) = x \qquad P_3(x) = x - \frac{x^3}{3!}$

$p_1(x) = x \qquad p_3(x) = x - \frac{x^3}{3!}$

$p_2(x) = x \qquad p_5(x) = x - x^3/3! + x^5/5!$

$\longrightarrow |\sin x - p_1(x)| \leq ?$

$|\sin x - x| \leq ? \qquad -\pi/4 \leq x \leq \pi/4$

$$|\sin x - p_n(x)| = \left| \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c_x) \right|$$

$$\left|\frac{d}{dx}(\sin x)\right| = |\cos x| \rightarrow \left|\frac{d}{dx}(\cos x)\right| = |-\sin x| \quad \sin x$$

$f(x) = \sin x$

$$\left| \frac{d^{n+1}}{dx^{n+1}} f(x) \right| = \begin{cases} |\sin x| & \text{if } n+1 \text{ is even} \\ |\cos x| & \text{if } n+1 \text{ is odd} \end{cases}$$

$$|\sin x - p_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} * |f^{(n+1)}(c_x)| \qquad -\pi/4 \leq x \leq \pi/4$$

$$\leq \frac{(\pi/4)^{n+1}}{(n+1)!}$$

??

n+1 is even

$c_x = \pi/4$

$|\sin c_x|$

$\sqrt{2}/2$

n+1 is odd!

$c_x = 0$

$|\cos c_x|$

1

$$= \frac{(\pi/4)^{n+1}}{(n+1)!} \max\{\sqrt{2}/2, 1\}$$

$$\leq \frac{(\pi/4)^{n+1}}{}$$

$$\leq \frac{(\pi/4)^{n+1}}{(n+1)!}$$