Tuesday, January 21, 2020 4:48 PM

e^{-1}

$\exp(x)$ $f(x)=e^x$

$\approx$ APPROXIMATELY

e^{-1}

$e \approx 3.14\ldots$

$e^{-1} = 1/e \approx 1/3$

$f(x)$ replaced by simple function simple Taylor poly.

Taylor series about a:

$$f(x) = \underbrace{f(a) + f'(a)(x-a)} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \ldots$$

$$= \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}$$

Taylor poly (simple function) of degree 1

$P_1(x) = f(a) + f'(a)(x-a)$

evaluate $f(x) = e^x$ at $x=-1$.

Taylor poly for e^x about $a=0$:

$p_1(x) = f(0) + f'(0)(x-0)$

$f(x) = e^x$ $f'(x) = e^x$, $f''(x) = e^x \ldots$

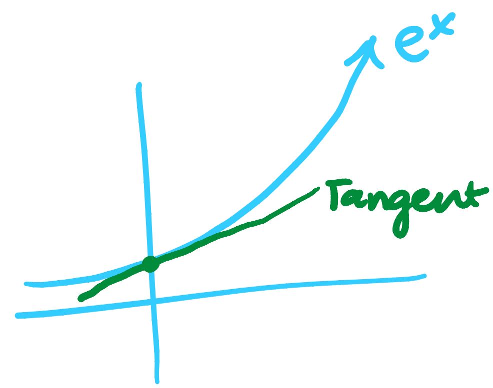
$p_1(x) = e^0 + e^0 x = 1+x$

$\hookrightarrow p_1(x) = e^0 + e^0 x = 1 + x$

Complicated functn $\quad e^x \approx p_1(x) = 1 + x \rightarrow$ Taylor poly of deg 1.

$e^{-1} \approx p(-1) = 1 - 1 = 0$

$$p_1(x) = \underbrace{f(0) + f'(0)(x-0)}_{\text{formula tangent at } (0, f(0))}$$



approximate e^{-1} by simplee function

$$f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)(x-0)^2}{2!}$$

$f(x) = e^x \quad f'(x) = f''(x) = e^x$

$$\hookrightarrow f(x) = e^0 + e^0 x + e^0 \frac{x^2}{2}$$

$$= 1 + x + \frac{x^2}{2} = p_2(x)$$ Taylor poly of degree 2 about 0

$e^{-1} = 0.3678$

$f(-1) \nearrow$ True value

$$p_2(-1) = \underbrace{1 + (-1) + \frac{(-1)^2}{2} = 0.5}_{\text{approximated value}}$$

True value − approximated value

approximated value

error = True value − approximated value

$= -0.1321$

compare this to error $= 0.3678 - p_1(-1) = 0.3678$

$$\int e^{\sin \pi x}\,dx \approx \boxed{\int p_n(x)\,dx}$$

Taylor poly of deg. n

easier than $\int e^{\sin \pi x}\,dx$

check: $f(-0.5) \approx p_1(-0.5)$

$p_2(-0.5)$

Error Representation:

Without calculating $p_3(x)$, give an estimate for $|f(x) - p_3(x)| \le ?$

where $f(x) = e^x \cos x$

$p_3(x) =$ Degree 3 Taylor poly for $f(x)$ about 0.

and $-\pi \le x \le \pi$.

and $-\pi \le x \le \pi$.

use error Representation formula:

$$f(x) - p_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c_x)$$

$n=3$, $a=0$, c_x is unknown $f(x)=e^x\cos x$.

$$f(x) - p_3(x) = \frac{x^4}{4!} f^{(4)}(c_x)$$

c_x is unknown number bet. x & 0.

$$f(x) = e^x \cos x$$

$$f'(x) = \frac{d}{dx}\left(e^x \cos x\right)$$ Product Rule for Differentiation

$$= \frac{d}{dx}(e^x) \cos x + e^x \frac{d}{dx}(\cos x)$$

$$= e^x \cos x + e^x(-\sin x)$$

$$= e^x(\cos x - \sin x) = e^x \cos x - e^x \sin x$$

$$f''(x) = \frac{d}{dx}(e^x \cos x) - \frac{d}{dx}(e^x \sin x)$$

already ?

already know

$e^x(\cos x - \sin x)$

$$\frac{d}{dx}(e^x \sin x) = \frac{d}{dx}(e^x)\sin x + e^x \frac{d}{dx}(\sin x)$$

$$= e^x \sin x + e^x \cos x$$

$$f''(x) = e^x(\cos x - \sin x) - \frac{d}{dx}(e^x \sin x)$$

$$= e^x(\cos x - \sin x) - (e^x \sin x + e^x \cos x)$$

$$= e^x \cancel{\cos x} - e^x \sin x - e^x \sin x - e^x \cancel{\cos x}$$

$$= -2e^x \sin x$$

$$f'''(x) = \frac{d}{dx}\left(-2e^x \sin x\right) = -2\frac{d}{dx}(e^x \sin x)$$

$$= (-2)\left(e^x \sin x + e^x \cos x\right)$$

$$f''''(x) = \frac{d}{dx}\left(-2(e^x \sin x + e^x \cos x)\right)$$

$$f''''(x) = -4e^x \cos x$$

Back to problem:

$$f(x) - p_3(x) = \frac{x^4}{4!} f^{(4)}(c_x) \qquad -\pi \le x \le \pi$$

$f(x) - p_{3}(x) = \frac{x^4}{4!} f^{(4)}(c_x)$

$$\left| f(x) - p_3(x) \right| = \left| \frac{x^4}{4!} \left(-4 e^{c_x} \cos c_x\right) \right|$$

c_x is unknown bet. x & 0.

Simplify $4! = 1 * 2 * 3 * 4$

$$= \left| \frac{x^4}{\cancel{4} * 3 * 2} * (-\cancel{4}) e^{c_x} \cos c_x \right|$$

$$= \frac{1}{6} \left| x^4 e^{c_x} \cos c_x \right| \qquad -\pi \le x \le \pi.$$

$$= \frac{1}{6} \underbrace{\left| x^4 \right|} * \left| e^{c_x} \cos c_x \right|$$

$-\pi \quad 0 \quad \pi$

$|x| \le \pi$

$|x|^4 \le \pi^4$

$x = \pi$

$$\le 1/6 \; \pi^4 * \left| e^{c_x} \cos c_x \right|$$

c_x → $-\pi \le c_x \le \pi$

$c_x = \pi$

$$\le \pi^4/6 * \left| e^{\pi} \cos \pi \right|$$

$$= \pi^4 e^{\pi} * |-1| = \underline{\pi^4 e^{\pi}}$$

$$= \frac{\pi^4}{6} e^{\pi} * |-1^{\psi}| = \frac{\pi^4 e^{u}}{6}$$