Math 4329: Worksheet 06
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Name: ____

1. Consider the fixed point iteration

$$x_{n+1} = 5 - (4+c)x_n + cx_n^5. \tag{1}$$

For some values of c, the iterations generated by the above formula converges to $\alpha = 1$ provided x_0 is chosen sufficiently close to α.

   (a) Identify the function $g(x)$ which characterizes the above fixed point iteration (1). [That is, the function $g(x)$ satisfying $x_{n+1} = g(x_n)$.]

   (b) Find the values of c to ensure the convergence of the iterations generated by the above formula provided x_0 is chosen sufficiently close to α.

   (c) For what values of c is this convergence quadratic?

2. Consider the task of finding a root $\alpha \approx 1.2564$ of the following equation

$$f(x) := e^x - 2x - 1 = 0, \quad x \in [1,2]. \tag{2}$$

We consider the following three fixed point iterative methods **Iter_1**–**Iter_3** to solve (2):

Iter_1: $x_{n+1} = \frac{e^{x_n}-1}{2}$

Iter_2: $x_{n+1} = e^{x_n} - x_n - 1$

Iter_3: $x_{n+1} = \ln(2x_n + 1)$.

Each of the iteration formulas **Iter_1**–**Iter_3** have the form

$$\boxed{x_{n+1} = g(x_n)}$$

for appropriately chosen continuous functions $g(x)$.

   (a) Determine (without actually iterating the formulas) which of the fixed point iterations **Iter_1**–**Iter_3** will converge to the root α (provided the initial guess x_0 is chosen to be sufficiently close to α).

   Furthermore, show that the fixed point iterative methods which converge, do so at a linear rate.

   (b) Design a fixed point iterative method which converges **quadratically** (provided the initial guess x_0 is chosen sufficiently close to α) and assumes the following form:

$$\boxed{\textbf{Iter\_4:} \quad x_{n+1} = g_4(x_n)}$$

for a suitable choice of $g_4(x)$. Please specify the function $g_4(x)$ characterizing the iterative method and provide sufficient reason for the quadratic convergence of **Iter_4**.

(c) Write a MATLAB program that approximates the root α in the interval $[1, 2]$ using the iterative method **Iter_4** that you obtained in part (b). Call this program `yourlastname_iter4.m`.
Please make sure that your program takes as **input**:

i. $\mathtt{x}_0$: initial guess
ii. `tol`: error tolerance
iii. `nmax`: maximum number of permissible iterations.

Make sure that the **output** printed to the screen looks like:

```
-------------------------------------------
 k  x_k   f(x_k)   \alpha - x_k     order
-------------------------------------------
 0
 1
 2
 .
 .
 .
  ------------------------------------------
```

where order $\approx \ln|\frac{(x_{k+1}-x_k)}{(x_k-x_{k-1})}|/\ln|\frac{(x_k-x_{k-1})}{(x_{k-1}-x_{k-2})}|$, $k \geq 2$.