Math 4329: Numerical Analysis Chapter 04: Spline Interpolation

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Why another interpolating polynomial?

Consider the following discrete data:

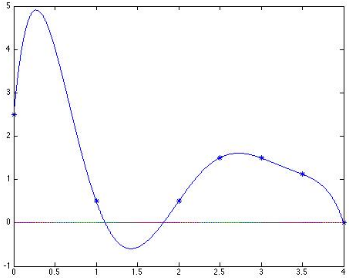
x	0	1	2	2.5	3	3.5	4
y	2.5	0.5	0.5	1.5	1.5	1.125	0

Our goal is to construct a polynomial which:

1. interpolates the given 7 data points,
2. has range between 0 and 2.5,
3. does not contain sharp corners i.e., a smooth function.

Idea

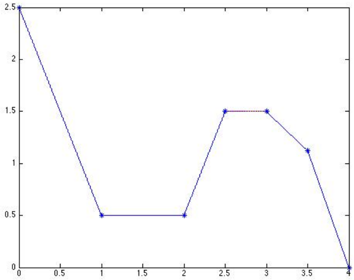
We can construct a polynomial interpolating 7 points. This polynomial should be of degree 6 and assumes the following shape



Voilates condition 2!

Idea

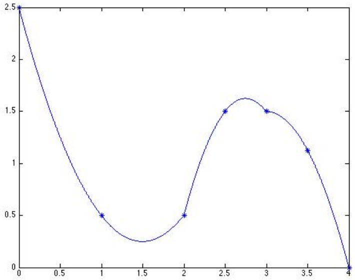
We can construct a piecewise linear polynomial simply by connecting the points by straight lines between $\{0, 1\}$, $\{1, 2\}$, $\{2, 2.5\}$, $\{2.5, 3\}$, $\{3, 3.5\}$, *and* $\{3.5, 4\}$.



Voilates condition 3!

Idea

Connect the data using a succession of quadratic interpolating polynomials for the following discrete data points:
$\{0, 1, 2\}$, $\{2, 2.5, 3\}$, *and* $\{3, 3.5, 4\}$.



Voilates condition 3 at x=2,3!

Conclusion

We need to construct an interpolating polynomial $s(x)$ which satisfies conditions (1)–(5)

$$s(x) \text{ is a cubic polynomial on } [x_{i-1}, x_i], \quad i = 1, 2, \cdots n, \tag{1}$$

$$s(x_i) = y_i \qquad i = 0, 1, \cdots n, \tag{2}$$

$$\lim_{x \to x_i^-} s'(x_i) = \lim_{x \to x_i^+} s'(x_i), \quad i = 1, \cdots n-1, \tag{3}$$

$$\lim_{x \to x_i^-} s''(x_i) = \lim_{x \to x_i^+} s''(x_i), \quad i = 1, \cdots n-1. \tag{4}$$

$$s''(x_0) = s''(x_n) = 0. \tag{5}$$

<u>Note:</u> *$s(x)$, $s'(x)$ and $s''(x)$ are continuous on $[x_0, x_n]$.*

Back to our original problem...

Calculate the natural cubic spline interpolating the data:

x	0	1	2	2.5	3	3.5	4
y	2.5	0.5	0.5	1.5	1.5	1.125	0

Using (1)–(5), we can construct the following cubic spline:

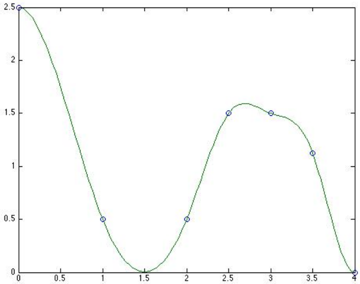


Figure : Satisfies the three conditions!

Questions on cubic splines

1. Find the cubic spline satisfying

$$s(0) = 0,\ s(1) = 1,\ s(2) = 2, s'(0) = 0,\ s''(2) = 2.$$

2. Check whether the following function is a spline:

$$s(x) = \begin{cases} x^3 & 0 \le x \le 1, \\ 2x - 1 & 1 < x < 2, \\ 3x^2 - 2 & 2 \le x \le 3. \end{cases}$$

3. Find a, b, c and d such that the following $s(x)$ is a natural cubic spline:

$$s(x) = \begin{cases} (x+1)^3, & -2 \le x \le -1, \\ ax^3 + bx^2 + cx + d, & -1 < x < 1, \\ (x-1)^2, & 1 \le x \le 2. \end{cases}$$

Construction of Splines

Introduce variables $M_0, \cdots, M_n$ such that

$$M_i \equiv S''(x_i), \quad i = 0, \cdots n.$$

Since $S(x)$ is a cubic spline on $[x_{j-1}, x_j]$
$\implies S''(x)$ is linear hence determined by its values at the end points x_{j-1} and x_j.

$$S''(x) = M_{j-1} \frac{x_j - x}{x_j - x_{j-1}} + M_j \frac{x - x_{j-1}}{x_j - x_{j-1}} \tag{6}$$

Construction of Splines

From the second antiderivative of $S(x)$ on $[x_{j-1}, x_j]$ and applying the interpolating conditions:
$S(x_{j-1}) = y_{j-1},\ S(x_j) = y_j$, we obtain

$$\begin{aligned} s(x) =& \frac{(x_j - x)^3 M_{j-1} + (x - x_{j-1})^3 M_j}{6(x_j - x_{j-1})} \\ &+ \frac{(x_j - x)y_{j-1} + (x - x_{j-1})y_j}{(x_j - x_{j-1})} \\ &- \frac{1}{6}(x_j - x_{j-1})\big((x_j - x)M_{j-1} + (x - x_{j-1})M_j\big), \end{aligned} \qquad (7)$$

where $j = 1, \cdots, n$.

Construction of Splines

Formula (6) ensures the continuity of $S''(x)$ while (7) implies the continuity of $S(x)$ and that it interpolates the given data. To guarantee the continuity of $S'(x)$ we require $S''(x)$ on $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ to have the same value at the knot $x_j,\ j = 1, \cdots n-1$.

$$\frac{x_j - x_{j-1}}{6} M_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} M_j + \frac{x_{j+1} - x_j}{6} M_{j+1} =, \quad \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}} \qquad (8)$$

$$M_0 = M_n = 0, \qquad j = 1, \cdots n-1. \qquad (9)$$

leads to the values of $M_0, \cdots, M_n$ and hence the spline $S(x)$.

Natural Spline Construction

Example

Calculate the natural cubic spline interpolating the data

$$\{(1,1),(2,\frac{1}{2}),(3,\frac{1}{3}),(4,\frac{1}{4})\}$$

Here, $n = 3$ and $x_{j+1} - x_j = 1$. The system in unknowns $M_0,\ M_1,\ M_2,\ M_3$ becomes:

$$\begin{aligned} \frac{1}{6}M_0 + \frac{2}{3}M_1 + \frac{1}{6}M_2 &= \frac{1}{3} \\ \frac{1}{6}M_1 + \frac{2}{3}M_2 + \frac{1}{6}M_3 &= \frac{1}{12}. \end{aligned}$$

Using $M_0 = M_3 = 0$ we obtain

$$M_1 = \frac{1}{2}, \quad M_2 = 0.$$

The spline is of the form:

$$s(x) = \begin{cases} \frac{x^3}{12} - \frac{x^2}{4} - \frac{x}{3} + \frac{3}{2}, & 1 \leq x \leq 2, \\ -\frac{x^3}{12} + \frac{3x^2}{4} - \frac{7x}{3} + \frac{17}{6}, & 2 \leq x \leq 3, \\ -\frac{x}{12} + \frac{7}{12}, & 3 \leq x \leq 4. \end{cases}$$

Error Analysis

So far we only interpolated data points, wanting a smooth curve. When we seek a spline to interpolate a known function, we are interested also in the accuracy.

Theorem

Let $f(x)$ be a function defined on $[a, b]$ that we want to interpolate on evenly spaced nodes/points $x_0, x_1, \cdots, x_n$.

$$h = \frac{b-a}{n}, \quad x_j = a + (j-1)h, \; j = 1, \cdots, n+1$$

and $s_n(x)$ be a natural cubic spline interpolating $f(x)$ at $x_0, \cdots x_n$. Then,

$$\max_{a \leq x \leq b} |f(x) - s_n(x)| \leq ch^2$$

where c depends on $f''(a)$ and $f''(b)$ and $\max_{a \leq x \leq b} |f^{(4)}(x)|$.