



Math 4329:
Numerical
Analysis
Chapter 04:
Polynomial
Interpolation

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Revisiting the function evaluation problem

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Most functions cannot be evaluated exactly:

$$\sqrt[2]{x}, e^x, \ln x, \text{trigonometric functions}$$

since by using a computer we are limited to the use of elementary arithmetic operations

$$+, -, \times, \div$$

With these operations we can only evaluate polynomials and fractions involving polynomials divided by polynomials (rational functions).



What does interpolation mean?

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Definition

In the mathematical field of numerical analysis, interpolation is a method of constructing new data points within the range of a discrete set of known data points.



Interpolation

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Given points

$$x_0, x_1, \dots, x_n$$

and corresponding values

$$y_0, y_1, \dots, y_n$$

find a function $f(x)$ such that

$$f(x_i) = y_i \quad i = 0, 1, \dots, n.$$

The interpolation function $f(x)$ is usually taken from a restricted class of functions: **polynomials**.



Interpolation of functions

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Given a function $f(x)$, and points

$$x_0, x_1, \dots, x_n$$

$$f(x_0), f(x_1), \dots, f(x_n)$$

find a polynomial or any other special function such that

$$p(x_i) = f(x_i), \quad i = 0, 1, \dots, n.$$

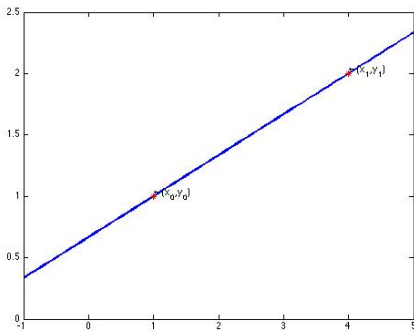
What is the error in approximating $f(x)$ by $p(x)$?



Linear Interpolation

Given two sets of points $\{(x_0, y_0), (x_1, y_1)\}$ with $x_0 \neq x_1$, draw a line through them, i.e. the graph of a linear polynomial

$$l(x) = \frac{x - x_1}{x_0 - x_1}y_0 + \frac{x - x_0}{x_1 - x_0}y_1$$





Example

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Find the polynomial $P_1(x)$ passing through $(1, 1)$ and $(4, 2)$.

$$P_1(x) = \frac{(4 - x) * 1 + (x - 1) * 2}{3}$$

Can you guess a function (besides the straight line) fitting the giving data?

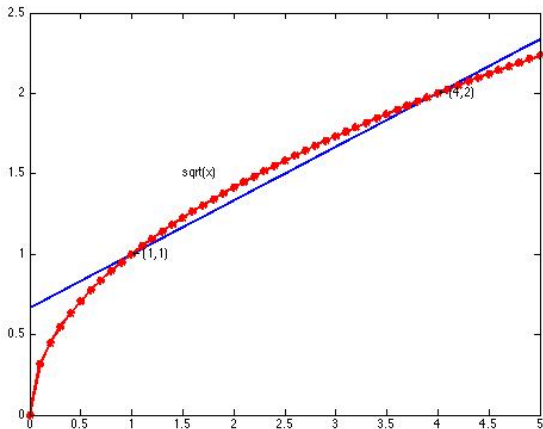


Example

The graph $y = P_1(x)$ and $y = \sqrt{x}$ from which the data is taken is:

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Remarks

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- The factor $\frac{(x-x_1)}{(x_0-x_1)}$ is 1 at $x = x_0$ and is 0 at $x = x_1$.
- This inspires us to be able to generalize the formula when we have three data points $\{(x_0, y_0), (x_1, y_1), (x_2, y_2)\}$ instead of two!

$$P_2(x) = y_0L_0(x) + y_1L_1(x) + y_2L_2(x)$$



Remarks

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Lagrange's Formula for interpolating polynomial

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$$P_2(x) = y_0L_0(x) + y_1L_1(x) + y_2L_2(x)$$

where

$$1 \quad L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$2 \quad L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \quad (\text{almost always has a negative sign!})$$

$$3 \quad L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Example

Find the polynomial interpolating the following data:
(1, 1), (4, 2), (9, 3).

$$p_2(x) = \frac{1}{24}(x-4)(x-9) - \frac{2}{15}(x-1)(x-9) + \frac{3}{40}(x-1)(x-4).$$

Check that $p_2(x)$ interpolates the given points!



Lagrange's Formula for interpolating polynomial

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Theorem

Given $n + 1$ data points

$$(x_0, y_0), (x_1, y_1) \cdots, (x_n, y_n)$$

with all x_i s being distinct, \exists **unique** $p_n(x)$ of degree at most n such that

$$p_n(x_i) = y_i, \quad i = 0, \cdots, n,$$

given by the formula

$$p_n(x) = \sum_{i=0}^n y_i L_i(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x).$$



Disadvantage

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Remark

The Lagrange's formula is suited for theoretical uses and when the number of discrete points is fixed, but is impractical for computing the value of an interpolating polynomial in the following sense: knowing $p_2(x)$ does not lead to a less expensive way to compute $p_3(x)$.



Towards a new interpolating polynomial formula

This necessitates a new way to construct polynomials that interpolate given discrete data points

$$(x_0, y_0), (x_1, y_1) \cdots, (x_n, y_n).$$

We need the notion of divided difference.

Set $f(x_i) = y_i$, $i = 0, \dots, n$.

Definition

For a pair of distinct points x_i, x_{i+1} , the **first-order Newton's divided difference** of $f(x)$ is given by

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$



Physical Significance of first-order Newton's Divided Difference

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If $f(x)$ is differentiable on an interval containing x_i and x_{i+1} , then the mean value theorem gives

$$f[x_i, x_{i+1}] = f'(c), \quad \text{where } c \text{ lies between } x_i \text{ and } x_{i+1}.$$

If x_i and x_{i+1} are close together, then

$$f[x_i, x_{i+1}] \approx f'\left(\frac{x_i + x_{i+1}}{2}\right)$$

is usually a very good approximation.

Example

For $f(x) = \cos(x)$, $x_0 = 0.2$, $x_1 = 0.3$,

$$f[x_0, x_1] = \frac{\cos(0.3) - \cos(0.2)}{0.3 - 0.2} \approx -0.2473009$$

$$f'\left(\frac{x_0 + x_1}{2}\right) = -\sin(0.25) \approx -0.247404.$$





Higher-order Divided Differences

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Suppose that x_0, x_1, \dots, x_n are distinct numbers. The **divided difference of order n** is defined as:

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, (n = 1)$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}, (n = 2)$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}, (n = 3).$$



Matlab function: `divdif()`

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We discuss a Matlab function:

```
divdif_y=divdif(x_nodes,y_values)
```

It calculates the divided differences of the function values given in the vector `y_values`, which are the values of some function $f(x)$ at the nodes given in `x_nodes`.

On exit,

$$\text{divdif_y}(i) = f[x_1, \dots, x_i], \quad i=1, \dots, m$$

with m the length of `x_nodes`.



divdif.m

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```
function divdif_y = divdif(x_nodes,y_values)

divdif y = y_values;
m = length(x_nodes);
for i=2:m
for j=m:-1:i
divdif y(j) = (divdif y(j)-divdif y(j-1)) ..
/(x_nodes(j)-x_nodes(j-i+1));
end
end
```



Newton's Divided Difference (D.D) Interpolation Formula

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We denote the polynomial generated by Newton's formula by the superscript N .

For the given data: $\{(x_0, y_0), (x_1, y_1)\}$, the interpolating polynomial is

$$P_1^N(x) = f(x_0) + f[x_0, x_1](x - x_0)$$

For $\{(x_0, y_0), (x_1, y_1), (x_2, y_2)\}$ the interpolating polynomial is

$$P_2^N(x) = P_1^N(x) + f[x_0, x_1, x_2](x - x_0)(x - x_1),$$

where $f(x_i) = y_i$, $i = 0, 1, 2$.



Remarks about Polynomial Interpolation

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- In general, the interpolation points x_i need not be evenly spaced, nor arranged in any order (increasing or decreasing).
- Practical Implementation: Efficient implementation can be realized by writing $P_2^N(x)$ as:

$$\begin{aligned} P_2^N(x) &= f(x_0) + f[x_0, x_1](x - x_0) + \\ &\quad f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &= f(x_0) + (x - x_0) \left(f[x_0, x_1] + f[x_0, x_1, x_2](x - x_1) \right) \end{aligned}$$

First, we calculate the nested term
 $\left(f[x_0, x_1] + f[x_0, x_1, x_2](x - x_1) \right)$
and then multiply the result with $(x - x_0)$.



Work Out Example

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Example

Using Newton's D.D. formula calculate the polynomial interpolating the following data:

$$\{(1, 1), (4, 2), (9, 3)\}$$

Show that it is the same polynomial obtained using the Lagrange's formula.

The two polynomials turn out to be the same! This is no coincidence since the polynomials interpolating the same set of data points is unique.



Error Formula for Newton's interpolating polynomial

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Theorem

*Let $P_n^N(x)$ denote the Newton's interpolating polynomial.
Suppose the $n + 1$ distinct points x_0, \dots, x_n are in the interval I .
Then,*

$$f(x) - P_n^N(x) = (x - x_0)(x - x_1) \cdots (x - x_n) f[x_0, x_1, \dots, x_n, x]$$

for any $x \in I$.

Notice that this error formula is impractical to use due to the appearance of x in the divided difference formula!



What is the advantage of the Lagrange's polynomial formula?

The Lagrange's polynomial formula is useful for **error analysis**.

Theorem

Let $P_n(x) = \sum_{j=0}^n f(x_j)L_j(x)$ denote the Lagrange interpolating polynomial. Suppose the $n + 1$ distinct points x_0, \dots, x_n are in the interval I . Then,

$$f(x) - P_n(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n) f^{(n+1)}(c_x)}{(n + 1)!} \quad (1)$$

for any $x \in I$, and c_x is an unknown between the minimum and maximum of x_0, \dots, x_n .

Notice the similarities with the error formula for the Taylor's polynomial by setting $x_i = a \quad \forall i = 0, \dots, n$.



Work-out Example

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Example

Without computing the interpolating polynomial $p(x)$, estimate the error in interpolating the $f(x)$ where

1 $f(x) = e^x$, at $0, 0.5, 1, -1$,

2 $f(x) = \sqrt{x}$. at $1, 4, 16, 9$.

Solution

For both the choices of functions $f(x)$,

- *there are 4 given points, the polynomial is of degree 3, i.e., $n = 3$ in the Lagrange error formula (1),*

- *Also,*

$$-1 \leq x \leq 1, \text{ for } e^x$$

- *Also,*

$$1 \leq x \leq 16, \text{ for } \sqrt{x}$$





Solution (continued)

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Solution

For $f(x) = e^x$,

$$|f(x) - p_3(x)| = \frac{|x(x-0.5)(x^2-1)|e^{cx}}{4!} \leq \frac{1*1.5*1*e^1}{24} \approx 0.17.$$

For $f(x) = \sqrt{x}$, $|f(x) - p_3(x)| =$

$$\frac{|(x-1)(x-4)(x-16)(x-9)|(15/16c_x^{-7/2})}{4!} \leq \frac{15*1}{(16*24)} \approx 0.04.$$