Math 4329: Numerical Analysis Chapter 03: Fixed Point Iteration and Ill behaving problems

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Why another root finding technique?

- Fixed Point iteration gives us the freedom to design our own root finding algorithm.
- The design of such algorithms is motivated by the need to improve the speed and accuracy of the convergence of the sequence of iterates $\{x_n\}_{n\geq 0}$.
- In this lecture, we will explore several algorithms for a given root finding problem and evaluate the convergence of each algorithm. Furthermore, we will look into the mathematical theory behind what makes certain methods converge.

Basic Idea Behind Fixed Point Iteration

- What is a fixed point?

  α is a fixed point of $g(x)$ provided $g(\alpha) = \alpha$.

  Here, α is being **"fixed"** by $g(x)$ since it maps it to itself.

- The root finding problem $\rightarrow$ fixed point finding problem.

$$f(x) = 0 \rightarrow \underbrace{f(x) + x}_{g(x)} = x$$

Towards the Design of Fixed Point Iteration

Consider the root finding problem

$$x^2 - 5 = 0. \qquad (*)$$

Clearly the root is $\sqrt{5} \approx 2.2361$.
We consider the following 4 methods/formulas M1-M4 for generating the sequence $\{x_n\}_{n \geq 0}$ and check for their convergence.

M1:

$$x_{n+1} = 5 + x_n - x_n^2$$

How? Multiply (*) by -1 and add x to both sides, then the root finding problem (*) is transformed into the problem of finding the root of

$$x = g(x) \text{ with } g(x) = x - x^2 + 5. \qquad (1)$$

Towards the Design of Fixed Point Iteration

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Consider the root finding problem

$$x^2 - 5 = 0. \qquad (*)$$

M2:

$$x_{n+1} = \frac{5}{x_n}$$

How? Add 5 to both sides of (*) and divide both sides by x , then the root finding problem (*) is transformed into the problem of finding the root of

$$x = g(x) \text{ with } g(x) = \frac{5}{x}. \qquad (2)$$

Towards the Design of Fixed Point Iteration

Consider the root finding problem

$$x^2 - 5 = 0. \tag{*}$$

M3:

$$x_{n+1} = 1 + x_n - \frac{x_n^2}{5}$$

How? Multiply (*) by -1, divide by 5 and add x to both sides, then the root finding problem (*) is transformed into the problem of finding the root of

$$x = g(x) \text{ with } g(x) = 1 + x - \frac{x^2}{5}. \tag{3}$$

Towards the Design of Fixed Point Iteration

Consider the root finding problem

$$x^2 - 5 = 0. \qquad (*)$$

M4:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right).$$

How? (Try it out yourself!)
The root finding problem (*) is transformed into the problem of finding the root of

$$x = g(x) \text{ with } g(x) = \frac{1}{2}\left(x + \frac{5}{x}\right). \qquad (4)$$

Towards the Design of Fixed Point Iteration

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Underlying Motivation for the algorithm design: $x = g(x)$.

Performance of the 4 methods

	M1	M2	M3	M4
n	$x_{n+1} : 5 + x_n - x_n^2$	$5x_n^{-1}$	$1 + x_n - \frac{x_n^2}{5}$	$\frac{x_n + 5x_n^{-1}}{2}$
0	2.5	2.5	2.5	2.5
1	1.25	2.0	2.25	2.25
2	4.6875	2.5	2.2375	2.2361
3	-12.2852	2.0	2.2362	2.2361
$x_n \to \alpha$	No	No	Yes	Yes

Transformation of the root finding to the fixed point finding problem

$$f(\alpha) = 0 \to \alpha = g(\alpha)$$

What makes the convergence possible?

Theorem

Assume $g(x)$ and $g'(x)$ are continuous for $c < x < d$ with the fixed point $\alpha \in (c, d)$. Suppose that

$$|g'(\alpha)| < 1,$$

then, any sequence $\{x_n\}_{n \geq 0}$ generated by $x_{n+1} = g(x_n)$ converges to α.

Exercise: Check which of the four methods satisfies the conditions for convergence.

Convergence criterea for the four methods

	M1	M2	M3	M4
g(x)	$5+x-x^2$	$5x^{-1}$	$1+x-\frac{x^2}{5}$	$\frac{x+5x^{-1}}{2}$
$g'(x)$	$1-2x$	$-5x^{-2}$	$\frac{1-2x}{5}$	$\frac{1-5x^{-2}}{2}$
$g'(\alpha)$	$1-2\sqrt{5}\approx -3.47$	-1	$\frac{1-2\sqrt{5}}{5}\approx 0.11$	0
$x_n \to \alpha$	No	No	Yes	Yes
$g''(\alpha)$				0.44
$x_n \to \alpha$	No	No	Linear	Quad.

Observe that M1 and M3 assume the following form:

M1: $x = x + c(x^2-5), \quad c=-1.$

M3: $x = x + c(x^2-5), \quad c=-1/5.$

Design of Iterative Methods

We saw four methods which derived by algebraic manipulations of $f(x) = 0$ obtain the mathematically equivalent form $x = g(x)$.
In particular, we obtained a method to obtain a general class of fixed point iterative methods namely:
Transformation of the root finding to the fixed point finding problem

$$f(x) = 0 \rightarrow x = \underbrace{x + cf(x)}_{g(x)}$$

where c is a parameter that we can choose to guarantee the convergence.

For what values of c do we have convergence?

Recall the root finding problem:

$$f(x) = x^2 - 5$$

and the corresponding fixed point problem is

$$x = g(x) \text{ with } g(x) = x + cf(x)$$

Using the convergence criteria $|g'(\alpha)| < 1$, we have

$$-1 < 1 + 2c\alpha < 1$$

which simplifies to

$$-0.4472 \approx -\frac{1}{\alpha} < c < 0.$$

M1: $x = x + c(x^2 - 5), \quad c = -1$ outside $(-1/\alpha, 0)$!.
M3: $x = x + c(x^2 - 5), \quad c = -1/5$ within $(-1/\alpha, 0)$!.
This explains why there is convergence for M3 but not M1.

Criterea for achieving higher order convergence

Theorem

Assume that g is conitnuously differentiable in an interval I_α containing the fixed point α and

$$g'(\alpha) = g''(\alpha) = 0 \cdots g^{(p-1)}(\alpha) = 0, \;\; p \geq 2.$$

Then, for x_0 close enough to α,

$$x_n \to \alpha$$

and

$$|\alpha - x_{n+1}| \leq c|\alpha - x_n|^p$$

i.e., convergence is of order p.

Remarks

There are a number of reasons to perform theoretical error analyses of numerical method. We want to better understand the method,

1. when it will perform well,
2. when it will perform poorly, and perhaps,
3. when it may not work at all.

With a mathematical proof, we convinced ourselves of the correctness of a numerical method under precisely stated hypotheses on the problem being solved. Finally, we often can improve on the performance of a numerical method.

Ill-behaving Problems

We will examine two classes of problems for which the numerical root finding methods do not perform well.

Often there is little that a numerical analyst can do to improve these problems, but one should be aware of their existence and of the reason for their ill-behavior.

We begin with functions that have a multiple root.

Ill-behaving Problems: Multiple roots

Definition

Multiple Roots The root α of $f(x)$ is said to be of multiplicity m if

$$f(x) = (x-\alpha)^m h(x), h(\alpha) \neq 0$$

for some continuous function $h(x)$ and positive integer m.

This means that

$$f(\alpha) = f'(\alpha) = \cdots f^{(m-1)}(\alpha) = 0, \quad f^{(m)}(\alpha) \neq 0.$$

Example 1:

$$f(x) = (x-1)^2(x+2)$$

has roots $\alpha = 1$ with multiplicity 2 and $\alpha = -2$ is a simple root (with multiplicity 1).

Ill-behaving Problems: Multiple roots

Example 2:

$$f(x) = x^3 - 3x^2 + 3x - 1$$

has roots $\alpha = 1$ with multiplicity 3 and

$$f(\alpha) = f'(\alpha) = f^{('')}(\alpha) = 0, \quad f^{(''')}(\alpha) \neq 0.$$

Example 3:

$$f(x) = x^2 \left[\frac{2\sin^2(\frac{x}{2})}{x^2} \right] = x^2 h(x)$$

has roots $\alpha = 0$ with multiplicity 2

Numerical Evaluation of Multiple Roots

1. When the Newton and secant methods are applied to the calculation of a multiple root , the convergence of $\alpha - x_n$ to zero is much slower than it would be for simple root.
2. There is a large interval of uncertainty as to where the root actually lies, because of the noise in evaluating $f(x)$.

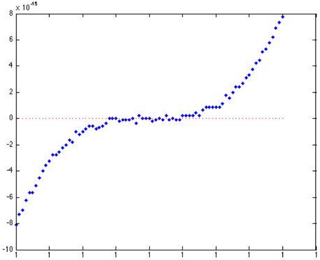

Figure : $f(x) = x^3 - 3x^2 + 3x - 1$ near $x = 1$.

Workout Example from Worksheet 05

Apply Newton's Method to $f(x) = -x^4 + 3x^2 + 2$ with starting guess $x_0 = 1$. Do we observe convergence?

Solution: No look at the sequence generated with the initial choice of x_0:

$$x_1 = -1, \quad x_2 = 1, \quad x_3 = 1, \quad x_4 = -1 \cdots .$$

What happens if we change the choice of x_0 to 0?

Solution: Since $f'(0) = 0$, we are unable to apply Newton's Method.

$$x_1 = -1 \quad x_2 = 1 \quad x_3 = 1 \quad x_4 = -1 \cdots$$

Workout Example from Worksheet 05

Apply Secant's Method to $f(x) = -x^4 + 3x^2 + 2$ with starting guess $x_0 = 0$ and $x_1 = 1$. Compute x_2 and x_3. Do we observe convergence?

Do it yourself in the class!

Workout Example from Worksheet 06

Consider the fixed point iteration

$$x_{n+1} = 5 - (4+c)x_n + cx_n^5. \tag{5}$$

For some values of c, the iterations generated by the above formula converges to $\alpha = 1$ provided x_0 is chosen sufficiently close to α.

1. Identify the function $g(x)$ which characterizes the above fixed point iteration (5). [That is, the function $g(x)$ satisfying $x_{n+1} = g(x_n)$.]
2. Find the values of c to ensure the convergence of the iterations generated by the above formula provided x_0 is chosen sufficiently close to α.
3. For what values of c is this convergence quadratic?