Numerical Analysis: Gaussian Numerical Integration

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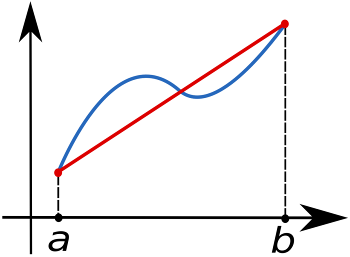
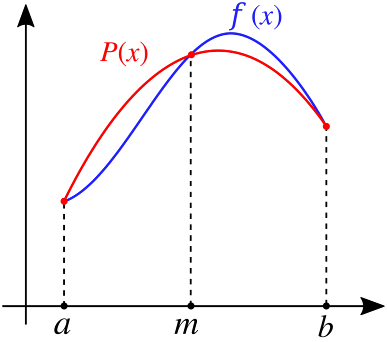
Notation

Traditionally, quadrature refers to area.
Numerical Integration rule is also called numerical quadrature rule.





General form of the integration rule

Towards designing a general integration rule, we first extract the general form of this rule:

$$I_n(f) = \sum_{j=1}^{n} \omega_j f(x_j)$$

where

- w_j denote the weights of the integration rule,
- x_i denote the nodes of the integration rule.

Let us see how this fits with the two integration rules we have learnt.

General form of the integration rule

The trapezoidal rule:

$$\int_a^b f(x)\, dx \approx \frac{b-a}{2}\Big[f(a) + f(b)\Big] \equiv T_1(f).$$

Weights: $w_1 = w_2 = \dfrac{b-a}{2}$,

Nodes: $x_1 = a, \ x_2 = b.$

$$T_1(f) = w_1 f(x_1) + w_2 f(x_2)$$

General form of the integration rule

The Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(a) + 4f(\frac{a+b}{2}) + f(b)\right] \equiv S_2(f).$$

Weights: $w_1 = h/3 \quad w_2 = 4h/3, \quad w_3 = h/3$

Nodes: $x_1 = a, \quad x_2 = \frac{a+b}{2}, \quad x_3 = b.$

$$S_2(f) = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

Designing a Numerical Integration Rule

Definition (Exactness of an integration formula)

Consider an integration formula

$$I_1(f) = w_1 f(x_1) + w_2 f(x_2).$$

This formula is said to be exact wrt $f(x)$ if

$$I(f) = I_1(f)$$

Designing a Numerical Integration Rule

Example

Consider approximating $I(f) = \int_0^1 f(x)\ dx$ by the Trapezoidal integration rule

$$T_1(f) \equiv \frac{1}{2}\Big[f(0) + f(1)\Big],$$

for any choice of $f(x)$.
We check which polynomial functions of the form $f(x) = 1,\ x,\ x^2, \cdots x^p,\ p > 0$ is this integration rule exact.

$$f(x) = 1,\ I(f) = \int_0^1 f(x)\ dx = 1 \quad T_1(f) = \frac{1}{2}\Big[f(0) + f(1)\Big] = 1.$$

So, the integration rule is exact for $f(x) = 1$.

Designing a Numerical Integration Rule

$$f(x) = x, \;\; I(f) = \int_0^1 f(x)\, dx = \frac{1}{2} \quad T_1(f) = \frac{1}{2}\Big[f(0) + f(1)\Big] = \frac{1}{2}$$

So, the integration rule is exact for $f(x) = x$.

$$f(x) = x^2, \;\; I(f) = \int_0^1 f(x)\, dx = \frac{1}{3} \quad T_1(f) = \frac{1}{2}.$$

So the integration rule is not exact for $f(x) = x^2$.

$T_1(f)$ is exact for polynomials of degree upto 1.

Designing a Numerical Integration Rule

To characterize the accuracy we demand from the integration rule, we introduce the notion of *degree of precision*.

Definition (Degree of Precision (DoP))

The degree of accuracy or precision of a quadrature/integration formula is the largest positive integer N such that the formula is exact for $1, \ x, \ x^2, \cdots x^N$.

Example

The trapezoidal rule has DoP 1.

Proof.

Refer to the previous example. □

Remark

The DoP of a quadrature formula is N if and only if the error is zero for all polynomials of degree $k = 0, \cdots N,$, but is NOT zero for some polynomial of degree $N + 1$.

Numerical Integration: A General Framework

Example

The Simpson's rule has DoP 3.

We observe that for

$$f(x) = 1,\ x,\ x^2,\ x^3\ I(f) = S_2(f).$$

However, $f(x) = x^4,\ I(f) = \int_0^1 f(x)\,dx = \frac{1}{5} = 0.2,$

$$\text{while,}\quad S_2(f) = \frac{h}{3}\Big[f(0) + 4f(0.5) + f(1)\Big],\ h = 1/2$$
$$= \frac{1}{6} \times 0 + \frac{4}{6} \times 0.5^3 + \frac{1}{6} \times 1 = 0.25.$$

$$I(f) = S_2(f) \quad \text{for} \quad f(x) = 1,\ x,\ x^2,\ x^3;$$
$$I(f) \neq S_2(f) \quad \text{for} \quad f(x) = x^4.$$

Table for Gaussian Quadrature

For another quadrature rule $I_n(f)$ to approximate $\int_{-1}^{1} f(x)\,dx$ of the form

$$I_n(f) = \sum_{i=1}^{n} w_i f(x_i)$$

we follow the weights and nodes given by the table:

n	x_i	w_i	n	x_i	w_i
2	± 0.57735	1	4	± 0.8611	0.3478
				± 0.33998	0.6521
3	± 0.77459	0.555	5	± 0.9061	0.2369
	0	0.8888		± 0.5384	0.4786
				0.0	0.5688

Table : n-point Gaussian Quadrature rule

Example

Example

Apply the 2 and 3 points Gaussian numerical integration formula to obtain an approximation $I_n(f)$ for $I = \int_{-1}^{1} e^x dx$.

Use the nodes and weights provided in Table.

Proof.

$$
\begin{aligned}
I_2(f) =& 1 \times e^{(-0.57735)} + 1 \times e^{(0.57735)}.\\
I_3(f) =& 0.555 \times e^{(-0.33998)} + 0.8888 \times e^{(0)} + \\
& 0.555 \times e^{(0.33998)}.
\end{aligned}
$$

□

Another example

Example

Apply the 2 and 3 points Gaussian numerical integration formula to obtain an approximation $I_n(f)$ for $I = \int_{-1}^{1} e^{-x^2} dx$.

Use the nodes and weights provided in Table.

Proof.

$$I_2(f) = 1 \times e^{-(-0.57735)^2} + 1 \times e^{-(0.57735)^2}.$$

$$I_3(f) = 0.555 \times e^{-(-0.33998)^2} + 0.8888 \times e^{-(0)^2} + 0.555 \times e^{-(0.33998)^2}.$$

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& 0.555 \times e^{-(0.33998)^2}.
\end{aligned}
$$

□

Designing Quadrature Rules

Example

Find c_1, and c_2 in the following quadrature formula:

$$\int_1^2 f(x)dx \approx c_1 f(1) \; + \; c_2 f(2) = \tilde{I}(f).$$

so that is exact for all polynomials of the largest degree possible. What is the degree of precision for this formula?

Designing Quadrature Rules

Proof.

$$f(x) = 1, \implies \tilde{I}(f) = c_1 + c_2,\ I(f) = \int_1^2 1\ dx = 1.$$

$$f(x) = x, \implies \tilde{I}(f) = c_1 + 2c_2,\ I(f) = \int_1^2 x\ dx = 3/2.$$

For the integration rule to be exact for $f(x) = 1$, $c_1 + c_2 = 1$.
Similarly, $c_1 + 2c_2 = 3/2$.
This means that $c_1 = c_2 = 1/2$.

DoP:

For $f(x) = x^2, \implies \tilde{I}(f) = c_1 + 4c_2 = 1/2 + 4/2 = 5/2$,

while $I(f) = \int_1^2 x^2\ dx = 7/3$.

Hence, $\tilde{I}(f) \neq I(f)$ for $f(x) = x^2$. Therefore DoP is 1. □