Math 4329: Numerical Analysis Chapter 03: Newton's Method

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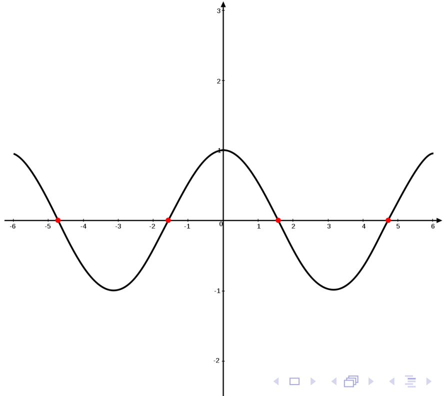
Mathematical question we are interested in numerically answering

- How to find the x-intercepts of a function $f(x)$? These x-intercepts are called the roots of the equation $f(x) = 0$. **Notation: denote the exact root by α. That means, $f(\alpha) = 0$.**



Basic Idea Behind Newton's Method

Given x_0, x_1 is the x-intercept of the tangent line at $(x_0, f(x_0))$.

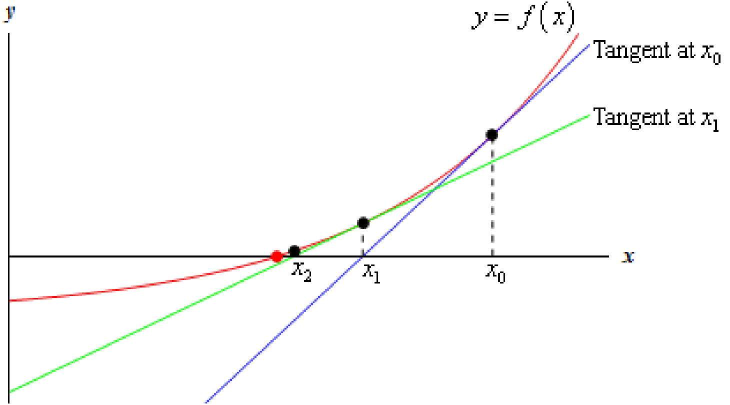


Figure : **Linearization of $f(x)$ about x_0, x_1 and x_2 respectively.**

Newton's Method

Tangent Line at $(x_0, f(x_0))$:

$$y(x) = f(x_0) + f'(x_0)(x - x_0).$$

We obtain the next iterate x_1 as the x-intercept of the tangent line that is

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

This simplifies to

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Generalizing, we can generate a sequence $\{x_n\}_{n \geq 1}$ where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots$$

Newton's Method

Let x_0 be an initial guess. Let $\varepsilon > 0$ denote the given error tolerance and `max_iteration` denote the permissible number of iterations.

If $|f(x_0)| \leq \varepsilon$, then accept x_0 as the root and stop.

Otherwise, define $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ and,

For $k = 1,\ 2,\ 3 \cdots,$ `max_iteration` **do**

- N1 If $|f(x_k)| \leq \varepsilon$ and $|x_k - x_{k-1}| < \varepsilon$ then accept x_k as the root and stop.
- N2 Define $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$.
- N3 Return to N1.

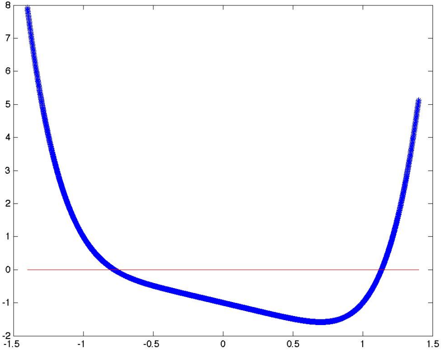
See the code Newton.m.

Example

Find the largest root of

$$f(x) = x^6 - x - 1 = 0$$

accurate within $\varepsilon = 1e - 8$ using Newton's Method.



Solution

Note $\alpha \approx 1.134724138$.

Solution: The sequence of iterates $\{x_n\}_{n\geq 1}$ is generated according to the formula:
for all $n = 0, 1, 2, \cdots$

$$
\begin{aligned}
x_{n+1} &= x_n - \left(\frac{x_n^6 - x_n - 1}{6x_n^5 - 1}\right), \\
&= x_n\left(\frac{6x_n^5 - 1}{6x_n^5 - 1}\right) - \left(\frac{x_n^6 - x_n - 1}{6x_n^5 - 1}\right) \\
&= \frac{6x_n^6 - x_n - \left(x_n^6 - x_n - 1\right)}{6x_n^5 - 1} \\
&= \frac{5x_n^6 + 1}{6x_n^5 - 1}.
\end{aligned}
$$

Performance of the Newton's Method

n	x_n	$f(x_n)$	$x_n - x_{n-1}$	$\alpha - x_{n-1}$
0	1.50	8.89e+1	–	–
1	1.30049088	2.5e+1	-2e-1	-3.65e-1
2	1.18148042	5.38e-1	-1.19e-1	-1.66e-1
3	1.13945559	4.92e-2	-4.2e-2	-4.68e-3
4	1.13477763	5.5e-4	-4.68e-3	-4.73e-3
5	1.13472415	7.11e-8	-5.35e-5	-5.35e-5
6	1.13472414	1.55e-15	-6.91e-9	-6.91e-9
⋮	⋮	⋮	⋮	⋮
α	1.134724138			

Remarks

1. May converge slowly at first. However, as the iterates come closer to the root, the speed of convergence increases.

Another Example

Using Newton's Method solve the following equation

$$f(x) \equiv x^3 - 3x^2 + 3x - 1 = 0$$

with an accuracy of $\varepsilon = 10^{-6}$.
Simplified form of Newton's Method:

$$x_{n+1} = \frac{2x_n^3 - x_n^2 + 1}{3(x_n - 1)^2},$$

with initial guess $x_0 = 0.5$.

Application I: Computing $a^{1/m}$

Compute $\sqrt{2}$ using only Newton's Method and '+,-,*,/'.

Solution: Find x such that $x^2 = 2$.
Equivalently, find x satisfying

$$f(x) := x^2 - 2 = 0$$

Newton's Method: Start with initial guess $x_0 = 1$, compute x_1 using

$$x_1 = x_0 - \frac{(x_0^2 - 2)}{2x_0} = 1.5$$

$$x_2 = 1.4166, \; x_3 = 1.4142, \; x_4 = 1.4142.$$

Application II: Division Operation

Replace the division operation in early computers. These early computers only allowed addition, subtraction and multiplication.

Compute $\frac{1}{b}$ using Newton's Method and the operations $+, -, *$.

Solution: Find x such that $x = \frac{1}{b}$.
Equivalently, find x satisfying

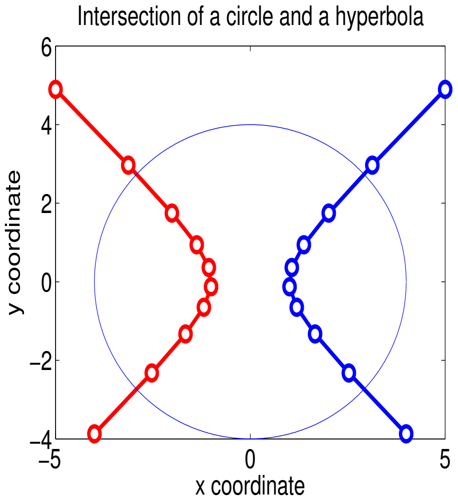
$$f(x) := b - x^{-1} = 0$$

Newton's Method: Start with initial guess x_0, compute x_1 using

$$x_1 = x_0(2 - bx_0).$$

Application III: Root finding in any dimension

Example: Finding the intersection of a hyperbola and a circle.



Error Analysis

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Assume that $f(x)$ has atleast continuous derivatives of order 2 for all x in some interval containing α and $f''(\alpha) \neq 0$.

$$\alpha - x_{n+1} = (\alpha - x_n)^2 \left[\frac{-f''(c_n)}{2f'(x_n)}\right].$$

Error in x_{n+1} is nearly proportional to the square of the error in x_n.
The term $\frac{-f''(c_n)}{2f'(x_n)}$ is the amplification factor. However, it depends on n. We need to make this factor independent of n. This can be achieved in the following manner:

$$\frac{-f''(c_n)}{2f'(x_n)} \approx \frac{-f''(\alpha)}{2f'(\alpha)} = M.$$

$$M = \max_{x \in [a,b]} \frac{-f''(x)}{2f'(x)}.$$

Error Analysis

Initial guess is crucial here and determine the number of iterations needed to achieve the desired accuracy!
For our worked out example,

$$\frac{-f''(c_n)}{2f'(x_n)} \approx \frac{-f''(\alpha)}{2f'(\alpha)} \approx -2.42.$$

$$\alpha - x_{n+1} \approx -2.42(\alpha - x_n)^2$$

Determining x_0 without using Bisection Method

$$\alpha - x_{n+1} = (\alpha - x_n)^2 \left[\frac{-f''(c_n)}{2f'(x_n)}\right]$$
$$\approx (\alpha - x_n)^2 \underbrace{\left[\frac{-f''(\alpha)}{2f'(\alpha)}\right]}_{M}$$

Multiplying both sides with M

$$M(\alpha - x_{n+1}) \approx M^2(\alpha - x_n)^2$$

$$M(\alpha - x_2) \approx M^2(\alpha - x_1)^2 \approx M^2\Big(M^2(\alpha - x_0)^4\Big)$$
$$= \Big(M(\alpha - x_0)\Big)^{2^2}.$$

$$|M(\alpha - x_0)| < 1 \implies |\alpha - x_0| < \frac{1}{|M|}$$

By picking x_0

$$-1 < \frac{1/b - x_0}{1/b} < 1$$

$$-1 < \frac{1 - bx_0}{1} < 1$$

$$0 < bx_0 < 2$$

Order of Convergence

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A sequence $\{x_n\}_{n\geq 0}$ converges to α with order $p \geq 1$

if $|\alpha - x_{n+1}| \leq c|\alpha - x_n|^p,\ n \geq 0$

for some $c \geq 0$

$p = 1$ and $c < 1$ linear convergence (Bisection Method),

$p = 2$ quadratic convergence (Newton's Method),

$p = 3$ cubic convergence (some fixed point iterative methods).