FINAL EXAM REVIEW

Thursday, December 5, 2019 1:31 PM

#1

$$\begin{bmatrix} 2 & \overset{\to \text{Row2}}{\boxed{-3}} & 0 \\ 1 & 3 & \underset{\text{Row3}}{\boxed{-10}} \\ \underset{\text{Row1}}{\boxed{3}} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{3} & & \\ & \boxed{-3} & \\ & & \boxed{-10} \end{bmatrix}$$

Row 3 <—> Row 1

$$\begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & -10 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \\ -7 \end{bmatrix}$$

$3 = |a_{11}| > |a_{12}| + |a_{13}| = 1$

Row 2 & Row 3 should be switched.

$$\begin{bmatrix} 3 & 0 & 1 \\ 2 & -3 & 0 \\ 1 & 3 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ -7 \\ 9 \end{bmatrix}$$

$|a_{22}| = |-3| = 3 > 2+0 = |a_{21}| + |a_{23}|$ ✓

$|a_{33}| = |-10| = 10 > 1+3 = 4$ ✓

then GS: with $X^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

apply one step GS: with $X^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$X^{(1)} = \begin{bmatrix} 13/3 \\ 7/3 \\ -10/9 \end{bmatrix}$$

2.) $$\cos x = \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots (-1)^n \frac{x^{2n}}{(2n)!}}_{p_{2n}(x)} + (-1)^{n+1} \frac{x^{2n+2} \cos c}{(2n+2)!}$$

find how large 2n needs to be so that

$$|\cos x - p_{2n}(x)| < 10^{-2}$$

$$\cos x = p_{2n}(x) + \frac{(-1)^{n+1}}{(2n+2)!} \cos c \; x^{2n+2} \qquad 0 \le x \le \pi$$

$$|\cos x - p_{2n}(x)| = \left| \frac{(-1)^{n+1}}{(2n+2)!} \cos c \; x^{2n+2} \right|$$

$$= \frac{1}{(2n+2)!} |\cos c| \, |x|^{2n+2} \qquad 0 \le x \le \pi$$

$c = 0$

0 ≤ x ≤ π

$|x| \le \pi$

$$\le \frac{1}{(2n+2)!} |\cos 0| \, \pi^{2n+2}$$

$$= \frac{\pi^{2n+2}}{(2n+2)!} \overset{?}{<} 10^{-2}$$

find n : $$\frac{\pi^{2n+2}}{(2n+2)!} < \frac{1}{100}$$

find n : $\dfrac{\pi}{(2n+2)!} < \dfrac{1}{100}$

check $n=5 \Rightarrow 2n+2 = 12$

$$\frac{\pi^{12}}{12!} = 0.00192 < 0.01$$

Note: $n=4$ doesnot give an error smaller that 0.01.

Answer: Poly deg should be 10.

#3 $\dfrac{e^x - e^{-x}}{2x}$ x is very close to zero.

when $x \approx 0$ $\dfrac{e^0 - e^{-0}}{2*0} = \dfrac{1-1}{0}$ $\dfrac{0}{0}$ $\dfrac{f(x)}{g(x)}$

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{2x} = \lim_{x\to 0} \frac{\left(e^x - (-e^{-x})\right)}{2} \quad \left(\frac{f'(x)}{g'(x)}\right)$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x}}{2} \quad \text{(L'HOPITAL RULE)}$$

$$= \frac{e^0 + e^{-0}}{2} = 1$$

Use the $\underbrace{\text{linear Taylor poly}}$ to avoid loss of significance.

e^x $\quad$ e^{-x} in $\dfrac{e^x - e^{-x}}{2x}$

$f(x) = e^x$

$$p(x) = f(0) + \frac{f'(0)(x-0)}{1!}$$

$$= f(0) + f'(0)\,x$$

$\boxed{1 - x = q(x)}$

$(1+x) - (1-x)$

$= f(0) + f'(0)\,x$

$= e^0 + e^0 x$

$\boxed{p(x) = 1+x}$

$$\frac{(1+x)-(1-x)}{2x}$$

$\downarrow$

$\frac{2x}{2x} = 1$

13 (e) Consider the roots of the equation (in x^{-1})

$x^{-2} + bx^{-1} + 1 = 0$ with $b > 0$

$y^2 + by + 1$, $y = x^{-1}$

The roots are:

$$x_1 = \frac{2}{-b+\sqrt{b^2-4}} \quad \& \quad x_2 = \frac{2}{-b-\sqrt{b^2-4}}.$$

Assume that b^2 is much larger than 4.

STATEMENT: x_2 will suffer from a loss of significance error. TRUE or false? FALSE ANSWER!

$$x_2 = \frac{2}{-b-\sqrt{b^2-4}}$$

$-b - \underbrace{\sqrt{b^2-4}}$

$-b - \approx b$

$\approx -2b$ (NO SUBTRACTION)

$$x_1 = \frac{2}{-b+\sqrt{b^2-4}}$$

$-b + \underbrace{\sqrt{b^2-4}}_{\approx b}$ LOSS OF SIG.

$\sqrt{x} - \sqrt{x-1}$ $x = 10^2, 10^3, 10^4 \ldots$

$$\sqrt{x} - \sqrt{x-1} \qquad x = 10,\ 10,\ 10 \ldots$$

$$\sqrt{b^2} - \sqrt{b^2-1}$$

4) $$f(x) = 1/_{3x+1} \qquad x_0 = 0,\ x_1 = 2,\ x_2 = 3.$$

(a) Calculate the piecewise linear poly. interpolating x_0, x_1, x_2

$$p(x) = \begin{cases} f(0) + f[\overset{x_0}{0}, \overset{x_1}{2}](x-0) & \text{on } [0,2] \text{ PIECE 1} \\ f(2) + f[2,3](x-2) & \text{on } [2,3] \text{ PIECE 2.} \end{cases}$$

x	$f(x)$	$f[x_i, x_{i+1}]$
0	1	$\dfrac{1/7 - 1}{2-0} = \dfrac{-6/7}{2} = \boxed{-3/7}$
2	$1/7$	$\dfrac{1/10 - 1/7}{3-2} = \boxed{-\dfrac{3}{70}}$
3	$1/10$	

$$p(x) = \begin{cases} 1 + (-3/7)\,x & \text{on } [0,2], \\ 1/7 + (-3/70)(x-2) & \text{on } [2,3]. \end{cases}$$

(b) find the quadratic interpolating polynomial with resp. to $(x_0, f(x_0))$, $(x_1, f(x_1))$ & $(x_2, f(x_2))$.

Answer: $p(x) = 1 - 3/7x + \frac{9}{70}x(x-2)$.

Check $p(0) = 1$ ✓

$p(2) = 1 - 6/7 = 1/7$ ✓

$p(3) = 1 - 9/7 + 27/70 = \frac{-20+27}{70}$

$= 1/10$

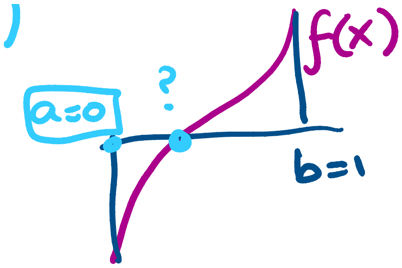
10) Root finding Techniques

$$x e^x = \cos x$$

find an initial interval $[a,b]$ which contains (x-values not f(x) values!) the smallest positive Root.

(x-values not f(x) values!)

the smallest positive Root.



$$f(x) = xe^x - \cos x$$

$f(0) = 0e^0 - \cos 0 = -1$

$f(1) = e - \cos 1 = 1.7184 > 0$

$f(\pi/2) = \pi/2\, e^{\pi/2} - \underbrace{\cos \pi/2}_{0} > 0$

$= \pi/2\, e^{\pi/2}$

> 0

$$\boxed{a = 0 \quad b = 1}$$

estimate # of midpoints c_n to be computed so that $|\alpha - c_n| < 10^{-9}$

$\alpha \rightarrow$ smallest pos. root. you can use:

$$|\alpha - c_n| \leq \frac{b-a}{2^n}.$$

$a=0,\ b=1$

find n such that

$$\frac{1-0}{2^n} < 10^{-9}$$

$$1/2^n < \frac{1}{10^9}$$

$$10^9 < 2^n$$

$n=10 \Rightarrow 2^n > 1000$

$$\ln(10^9) < \ln(2^n)$$

$$9\ln 10 < n \ln 2$$

$$29.897 \approx \frac{9\ln 10}{\ln 2} < n$$

$$30 < n \Rightarrow n \geq 31$$

(14)

Approx.

$f'(0.5)$

by

$D_h^+ f(0.5)$

$D_h^- f(0.5)$ where $h = 0.1, 0.2$.

x	f(x)
0.3	7.3891
0.4	7.4633
0.5	7.5383
0.6	7.6141
0.7	7.6906

$\underline{h=0.1}$

$$\begin{cases} D_h^+ f(0.5) = \dfrac{f(0.6) - f(0.5)}{0.1} \\[2ex] D_h^- f(0.5) = \dfrac{f(0.5) - f(0.4)}{0.1} \end{cases}$$

$h = 0.2$

$$D_h^+ f(0.5) = \frac{f(0.7) - f(0.5)}{0.2} = f[0.5, 0.7]$$

FYI

$$D_h^- f(0.5) = f(0.5) - f(0.3) = f[0.3, 0.5]$$

$$D_h^- f(0.5) = \frac{f(0.5) - f(0.3)}{0.2} = f[0.3, 0.5]$$

$$D_h^{(2)} f(x_1) = \frac{f(x_2) - 2f(x_1) + f(x_0)}{h^2} \approx f''(x_1)$$

$$D_h^{(2)} f(x_1) = \frac{D_h^+ f(x_1) - D_h^- f(x_1)}{h} \approx f''(x_1)$$

(a) $h = 0.1$ $\quad D_h^+ f(0.5) = \frac{f(0.6) - f(0.5)}{0.1} = 0.758$

(b) $h = 0.2$ $\quad D_h^{(2)} f(0.5) = ?$ Based on forward ($D_h^+ f$) & Backward ($D_h^- f$) Diff. Operators

$$D_h^+ f(0.5) = \frac{7.6906 - 7.5383}{0.3} = 0.7615 \qquad D_h^- f(0.5) = \frac{7.5383 - 7.3891}{0.2} = 0.746$$

$$D_h^{(2)} f(0.5) = \frac{0.7615 - 0.746}{0.2} = \frac{0.0155}{0.2} = 0.0775$$

(c) $h = 0.2$ $\quad D_h^{(2)} f(0.5) = ?$ Based on the direct f-values.

$$D_h^{(2)} f(0.5) = \frac{7.6906 - (2 * 7.5383) + 7.3891}{0.2^2}$$

$$= 0.0775$$

Same as (b)!

#15 See Exam 02 solutions

#15 see Exam 02 solutions

#13 (a) False $\quad \lim_{x\to 1^-} f'(x) \neq \lim_{x\to 1^+} f'(x)$

(b) False $\quad x_1 = 0 - 1/2$

(c) False because 4 digits means misrepresentation of $\frac{801}{800}$ rendering the given linear system to be inconsistent!

(d) False because $|\cos x - p_3(x)| \leq \frac{|x|^4}{4!} * |\cos c|$

$$\leq \frac{(\pi/4)^4}{4!} * 1$$

$$= 0.0158 > 0.001.$$

(e) False (Reason presented previously)

#12 (a)

$$(1111\ \ 1111\ \ 101)_2$$

$$= 2045$$

#12 (b) figure out $e = ?$ $\quad e = 10.$

figure out $E = ?$ $\quad E - 1023 = 10 \Rightarrow E = 1033$

$$= 1024 + 8 + 1$$

$$= 2^{10} + 2^3 + 2^0$$

E											MANTISSA										
0	1	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	1	1	1	0	1

#11 Look at the exam 03 Review.

#10(a) Done in class

10(b) $x_{n+1} = \underbrace{3 - (2+c)x_n + c x_n^3}_{g(x_n)}$

$g'(x) = -2 - c + 3cx^2$

$g'(\alpha) = -2 + 2c \quad , \alpha = 1$

Convergence Condition: $-1 < g'(\alpha) < 1$

$-1 < -2 + 2c < 1$

$-1 < -2 + 2c \qquad\qquad -2 + 2c < 1$

$\Rightarrow \ 1/2 < c \qquad\qquad c < 3/2$

$$\boxed{1/2 < c < 3/2}$$

#9 (Done in true-false)

#8 Done in exam 03 solutions.

#7 $\int_{-1}^{1} f(x)\,dx = a f(-1) + b f(1) + c f'(0.5)$.

$f(x) = 1 \Rightarrow a + b + 0 = 2 \rightarrow (1)$

$f(x) = x \Rightarrow -a + b + c = 0 \rightarrow (2)$

$f(x) = x^2 \Rightarrow a + b + c = 2/3 \rightarrow (3)$

$+ c = 2/3$ or $c = 2/3 - 2$

$f(x) = x$ ...

(1) & (3) gives us $\quad 2 + c = 2/3$ or $c = 2/3 - 2$
$= -4/3$

(2) and $c = -4/3$ gives us: $\quad -a+b = 4/3$

so (1) $\Rightarrow \quad a+b \; -a+b = 2 + 4/3 = 10/3$

$2b = 10/3 \Rightarrow b = 5/3$

$b = 5/3$ and $a+b = 2 \Rightarrow \quad a = 2 - 5/3 = 1/3$

$\boxed{\text{Answer: } \quad a = 1/3 \quad b = 5/3 \quad c = -4/3}$

DoP = ? check if for $f(x) = x^3$, the above quad formula holds.

$f(x) = x^3 \Rightarrow \int_{-1}^{1} f(x)\,dx = 0$

and $\quad a f(-1) + b f(1) + c f'(0.5)$
$= 1/3(-1) + 5/3 - 4/3(3 * 0.5^2) \neq 0$

so DoP is 2 and no HIGHER.

#6 $\quad \|M\| = |\alpha| + |\beta| < 1 \Rightarrow$ convergence

#5 $\quad$ FORMULAS

$$I - T_4 = -\frac{h^2}{2}(b-a) f''(c)$$

$$I - S_4 = -\frac{h^4}{180}(b-a) f''''(c)$$

$h = \frac{b-a}{4}$, $b = 1, a = 0$.

c is an unknown number between 0 and 1.

First we need the derivatives of $f(x) = \sqrt{x}\, e^x$ upto fourth order.

$f(x) = x^{1/2} e^x$

$f'(x) = 1/2\, x^{-1/2} e^x + x^{1/2} e^x$

$$f'(x) = \frac{1}{2} x^{-1/2} e^x + x^{1/2} e^x$$

$$f''(x) = -\frac{1}{4} x^{-3/2} e^x + \frac{1}{2} x^{-1/2} e^x + \frac{1}{2} x^{-1/2} e^x + x^{1/2} e^x$$

$$\boxed{f''(x) = -\frac{x^{-3/2}}{4} e^x + x^{-1/2} e^x + x^{1/2} e^x}$$

$$f'''(x) = \frac{3}{8} x^{-5/2} e^x - \frac{x^{-3/2}}{4} e^x - \frac{1}{2} x^{-3/2} e^x + x^{-1/2} e^x + \frac{1}{2} x^{-1/2} e^x + x^{1/2} e^x.$$

$$= 3/8\, x^{-5/2} e^x - \frac{3}{4} x^{-3/2} e^x + \frac{3}{2} x^{-1/2} e^x + x^{1/2} e^x$$

$$f''''(x) = -\frac{15}{16} x^{-7/2} e^x + \frac{3}{8} x^{-5/2} e^x + \frac{9}{8} x^{-5/2} e^x - \frac{3}{4} x^{-3/2} e^x + \left(-\frac{3}{4}\right) x^{-3/2} e^x + \underbrace{\frac{3}{2} x^{-1/2} e^x + \frac{1}{2} x^{-1/2} e^x} + x^{1/2} e^x$$

$$= -\frac{15}{16} x^{-7/2} e^x + \frac{12}{8} x^{-5/2} e^x - \frac{6}{4} x^{-3/2} e^x + 2 x^{-1/2} e^x + x^{1/2} e^x$$

Bounds on the error:

for Trapezoidal:

$$|I - T_4| = \left| -\frac{0.25^2}{2} (1-0) f''(c) \right|$$

$$= \left| -\frac{0.25^2}{2} * \left(-\frac{c^{-3/2}}{4} e^c + c^{-1/2} e^c + c^{1/2} e^c \right) \right|$$

Take $c = 1$ gives us an upper bound.

$$\leq \frac{0.25^2}{2} * \left(\frac{1}{4} + 2 \right) e = \frac{0.25^2}{2} * \frac{9}{4} * e = 0.1911.$$

$$|I - S_4| = \left| -\frac{0.25^4}{180} (1-0) f''''(c) \right|$$

$$|I - S_4| = \left| \frac{\cdots}{180} \cdots \right|$$

$$= \frac{0.25^4}{180} * \left| e^c \left(-\frac{15}{16} c^{-7/2} + \frac{12}{8} c^{-5/2} - \frac{6}{4} c^{-3/2} + 2c^{-1/2} + c^{1/2} \right) \right|$$

Again, take $c=1$ gives an upper bound.

$$\leq \frac{0.25^4}{180} * e * \left(-\frac{15}{16} + \frac{3}{2} - 2/3 + 3 \right)$$

$$= \frac{0.25^4}{180} * e * \left(\frac{-15}{16} + \frac{23}{6} \right)$$

$$= 1.7083 * 10^{-}$$