M4329

Thursday, November 21, 2019 7:11 AM

1(a) False

$$x + y = 0$$
$$x + \frac{401}{400} y = 1$$

3 digits means $\frac{401}{400} = 1.025$ represented by 1.02.

$\Rightarrow x = -y = 400$ fails to satisfy the above pair of equations.

1(b) TRUE The given iterative method can be expressed in form

$$N x^{(k+1)} = b + P x^{(k)}$$

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} -4c & c \\ c & 4c \end{bmatrix}.$$

Convergence guaranteed if $\|N^{-1}P\| < 1$

which holds if $5|c| < 1$

or $-\frac{1}{5} < c < 1/5$

$\Rightarrow \quad -0.2 < c < 0.2$

method Diverges if $|c| \geq 0.2$.

2(a)

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 5 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $\quad m_{21} = \frac{a_{21}}{a_{11}} = 1.$

$R_3 \to R_3 - 3R_1$

$m_{31} = \frac{a_{31}}{a_{11}} = 3$

$$\begin{array}{c} R_1 \end{array} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \end{array}\right.$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc:c} 2 & 1 & 1 & 1 \\ 0 & 5 & 7 & 2 \\ 0 & 5 & 15 & 2 \end{array}\right]$$

$$R_3 \to R_3 - R_2 \,, \quad m_{32} = 1$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc:c} 2 & 1 & 1 & 1 \\ 0 & 5 & 7 & 2 \\ 0 & 0 & 8 & 0 \end{array}\right]$$

Time for Back substitution:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ 5x_2 + 7x_3 &= 2 \\ 8x_3 &= 0 \end{aligned}$$

$$Ux = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$x_3 = 0$ in $5x_2 + 7x_3 = 2$ gives:

$$x_2 = 2/5$$

$x_3 = 0$, $x_2 = 2/5$ in $2x_1 + x_2 + x_3 = 1$ gives

$$\begin{aligned} 2x_1 + 2/5 + 0 &= 1 \\ 2x_1 &= -2/5 + 5/5 = 3/5 \\ x_1 &= 3/10 \end{aligned}$$

$\begin{bmatrix} 3/10 \\ \end{bmatrix}$ is the solution.

$$\begin{bmatrix} 3/10 \\ 2/5 \\ 0 \end{bmatrix} \text{ is the solution.}$$

2(b) LU Decomposition

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 8 \end{bmatrix}}_{U}$$

3)

$$A = \begin{bmatrix} 39 & 40 \\ 40 & 41 \end{bmatrix} \Rightarrow \|A\| = 81$$

$$A^{-1} = \begin{bmatrix} -41 & 40 \\ 40 & -39 \end{bmatrix} \Rightarrow \|A^{-1}\| = 81$$

$$\text{Cond}(A) = 81^2 = 6561 >> 1$$

$\Rightarrow$ any linear system $Ax = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ will be ill conditioned!

4

Jacobi: $N^J x^{(k+1)} = b + P^J x^{(k)} \quad k = 0, 1.$

4

Jacobi: $N^J x^{(k+1)} = b + P^J x^{(k)}$

$$N^J = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \qquad P^J = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

$$x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow x^{(1)} = \begin{bmatrix} -4/3 \\ 2/5 \end{bmatrix}$$

$$x_1^{(1)} = 1/3(-4) = -4/3$$

$$x_2^{(1)} = 2/5$$

$$x^{(1)} = \begin{bmatrix} -4/3 \\ 2/5 \end{bmatrix} \longrightarrow x^{(2)} = \begin{bmatrix} -18/15 \\ 14/15 \end{bmatrix}$$

$$3x_1^{(2)} = -4 + 2/5 = -18/5$$

$$5x_2^{(2)} = 2 - 2(-4/3) = \frac{6+8}{3} = \frac{14}{3}$$

Gauss Seidel:

$$x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow x^{(1)} = \begin{bmatrix} -4/3 \\ 14/15 \end{bmatrix}$$

$x_1^{(1)} \to$ same as Jacobi!

$$x_2^{(1)} = \frac{1}{5}(2 - 2x_1^{(1)})$$

$$= \frac{2}{5}\left(1 - (-4/3)\right) = \frac{2}{5} * \frac{7}{3} = \frac{14}{15}$$ same as $x_2^{(1)}$ of Jacobi!

$$x^{(1)} = \begin{bmatrix} -4/3 \\ \ldots \end{bmatrix} \longrightarrow x^{(2)} = \begin{bmatrix} -46/45 \\ \ldots \end{bmatrix}$$

$$X^{(1)} = \begin{bmatrix} -4/3 \\ 14/15 \end{bmatrix} \longrightarrow X^{(2)} = \begin{bmatrix} -46/45 \\ \frac{182}{225} \end{bmatrix}$$

$$x_1^{(2)} = \tfrac{1}{3}\left(-4 + \frac{14}{15}\right) = \tfrac{1}{3}\left(\frac{-60+14}{15}\right) = \frac{-46}{45}$$

$$x_2^{(2)} = \tfrac{1}{5}\left(2 - 2\left(-\frac{46}{45}\right)\right) = \frac{2}{5}\ \frac{91}{45} = \frac{182}{225}$$

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5) Option I: Done in text-book! Please show all details!

Option II: $p(x) = a_0 + a_1 x + a_2 x^2$

find a_0, a_1, a_2

$p(2)=4$ gives: $a_0 + 2a_1 + 4a_2 = 4$

$p(-1)=1$ gives: $a_0 - a_1 + a_2 = 1$

$p(1)=0$ gives: $a_0 + a_1 + a_2 = 0$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$b = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{AX = b}$$

$$\boxed{Ax = b}$$

5(b) Solve $Ax = b$ by solving:

$$Ly = b \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$$

$$Ux = y$$

Step 1: Write $A = LU$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow U$$

$R_2 \to R_2 - R_1 \qquad m_{21} = 1$

$R_3 \to R_3 - R_1 \qquad m_{31} = 1$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -3 \\ 0 & -1 & -3 \end{bmatrix}$$

$R_3 \to R_3 - m_{32} R_2, \quad m_{32} = \dfrac{a_{32}}{a_{22}} = 1/3$

$$U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -3 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/3 & 1 \end{bmatrix}$$

Solve $Ly = b$ and then

$Ux = y \rightarrow \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Solve for y_1, y_2, y_3:

$$\left.\begin{aligned} y_1 &= 4 \\ y_1 + y_2 &= 1 \\ y_1 + \frac{y_2}{3} + y_3 &= 0 \end{aligned}\right\} \text{RHS of } Ax=b.$$

$y_1 = 4 \Rightarrow y_2 = 1-4 = -3$ $\quad \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}$

$y_1 + y_2/3 + y_3 = 0 \Rightarrow 4 - 1 + y_3 = 0$

$y_3 = -3$

$$\underset{U}{\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -3 \\ 0 & 0 & -2 \end{bmatrix}} \underset{x}{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}} = \underset{y}{\begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}}$$

$$a_0 + 2a_1 + 4a_2 = 4$$
$$-3a_1 - 3a_2 = -3$$
$$-2a_2 = -3$$

$a_2 = 3/2$

$-a_1 - a_2 = -1 \Rightarrow a_1 = 1 - a_2 = 1 - 3/2 = -1/2$

$a_0 + 2a_1 + 4a_2 = 4$

$a_0 - 1 + 6 = 4 \Rightarrow a_0 = 1 - 6 + 4 = 1 - 2 = -1$

$$\boxed{p(x) = -1 - \frac{1}{2}x + \frac{3}{2}x^2}$$

$p(2) = 4$

$p(1) = 0$

$p(-1) = 1$