Math 1312 **Practice Exam 2**

There will be 6 or 7 problems on the actual test, plus one (challenging) extra credit problem. All of them will be similar to the problems shown here. (Note that in the given solutions, all work is **not** shown)

1) Find a formula for $a_n, n \geq 1$.

$$\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots$$

(The formula is $\{a_n\} = \left\{\frac{(-1)^{n+1}n^2}{n+1}\right\}$)

2) Find the exact sum of the series, if possible.

$$\sum_{n=3}^{\infty} \frac{2^n}{5^{n+1}}$$

The sum of the series can be found by rewriting the terms as $\frac{2^n}{(5)5^n} = \frac{1}{5}\left(\frac{2}{5}\right)^n$, so the series is

$$\frac{1}{5}\sum_{n=3}^{\infty}\left(\frac{2}{5}\right)^n = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^{n+3} = \left(\frac{1}{5}\right)\left(\frac{2}{5}\right)^3\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n = \left(\frac{8}{625}\right)\left(\frac{1}{1-\frac{2}{5}}\right) = \frac{8}{375}$$

3) Determine if the following series converge or diverge. You need to clearly state which test you are using, and show all of your work.

a) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+2}$ (use Limit Comparison Test, compare terms to $b_n = \frac{1}{n}$, and find that $\lim_{n\to\infty} \frac{n+1}{n^2+2} \cdot \frac{n}{1} = 1$, so series diverge together, and so given series diverges)

b) $\sum_{n=0}^{\infty} \frac{3^n}{n^3+3}$ (use Ratio Test, find $\lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^3+3} \cdot \frac{n^3+3}{3^n} = 3 > 1$, so series diverges)

c) $\sum_{n=1}^{\infty} \frac{n!}{(2n)!(n-1)!}$

(use Ratio Test, find $\lim_{n\to\infty} \frac{(n+1)!}{[2(n+1)]!n!} \cdot \frac{(2n)!(n-1)!}{n!} = 0 < 1$, so series converges)

d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^2+n+1}$

(use Alternating Series Test, state that $\lim_{n\to\infty} \frac{1}{2n^2+n+1} = 0$ and show that $\frac{1}{2(n+1)^2+(n+1)+1} < \frac{1}{2n^2+n+1}$, hence series converges)

e) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{e^n}$

(use Alternating Series Test, state that $\lim_{n\to\infty} \frac{1}{e^n} = 0$ and that $\frac{1}{e^{n+1}} < \frac{1}{e^n}$, so series converges)

f) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ (use Divergence Test, $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$ so series diverges)

g) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ (Use the integral test, writing $\lim_{b\to\infty} \int_2^b \frac{\ln x}{x} dx$. Since $\int \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2$, this limit is $\frac{1}{2} \lim_{b\to\infty} (\ln x)^2 |_2^b = \infty$, so integral is divergent, and thus the series is also divergent. Note that the comparison test could also be used here.)

4) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$$

(Given series converges by the AST, but $\sum_{n=1}^{\infty} \frac{1}{3n}$ diverges since it is a multiple of a p-series with $p = 1$, so series is conditionally convergent)

5) Find an expression for the general term of the following power series. Give the starting value for the index.

$$-\frac{(x-2)^2}{2} + \frac{(x-2)^4}{4} - \frac{(x-2)^6}{8} + \frac{(x-2)^8}{16} - \frac{(x-2)^{10}}{32} + \cdots$$

(General term is $\frac{(-1)^n (x-2)^{2n}}{2^n}$, and the starting value for the index is 1)

6) Find the 5th Maclaurin approximation for $f(x) = x \cos x$.

7) Use the ratio test to find the radius of convergence of the power series.

a) $\sum_{n=1}^{\infty} \frac{(n+2)x^n}{n}$ (Ratio Test gives $\lim_{n\to\infty} \left| \frac{(n+3)x^{n+1}}{n+1} \cdot \frac{n}{(n+2)x^n} \right| = |x| \lim_{n\to\infty} \frac{n^2+3n}{n^2+3n+2} = |x|$, so $|x| < 1$ means the radius of convergence is 1)

b) $\sum_{n=0}^{\infty} \frac{(2n)! x^n}{(n!)^2}$ (Ratio Test gives $\lim_{n\to\infty} \left| \frac{(2n+2)! x^{n+1}}{[(n+1)!]^2} \cdot \frac{(n!)^2}{(2n)! x^n} \right| = 2|x| \lim_{n\to\infty} \frac{2n+1}{n+1} = 4|x|$, so $4|x| < 1$ means $|x| < \frac{1}{4}$, so radius of convergence is $\frac{1}{4}$)

8) Find the radius and interval of convergence of the power series.

a) $\sum_{n=1}^{\infty} \frac{(x+1)^n}{3^n n^2}$

(Ratio Test gives $\lim_{n\to\infty} \left|\frac{(x+1)^{n+1}}{3^{n+1}(n+1)^2} \cdot \frac{3^n n^2}{(x+1)^n}\right| = \frac{1}{3}|x+1| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \frac{1}{3}|x+1|$, so $\frac{1}{3}|x+1| < 1$ means $|x+1| < 3$, so radius of convergence is 3. Since $|x+1| < 3$ means $-3 < x+1 < 3$, the interval of convergence (without the endpoints) is $-4 < x < 2$. Also it is necessary to find what happens at the endpoints of the interval, so plug $x = -4$ and $x = 2$ into the series. With $x = -4$, series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converges by the AST. If $x = 2$, the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series. Thus the interval of convergence is $[-4, 2]$)

b) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^{2n}}{4^n}$

(Ratio Test gives $\lim_{n\to\infty} \left|\frac{(x-3)^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{(x-3)^{2n}}\right| = \frac{1}{4}|x-3|^2$, and $\frac{1}{4}|x-3|^2 < 1$ means $|x-3| < 2$ so radius of convergence is 2. Since $|x-3| < 2$ means $-2 < x-3 < 2$, the interval of convergence (without the endpoints) is $1 < x < 5$. Also it is necessary to find what happens at the endpoints of the interval, so plug $x = 1$ and $x = 5$ into the series. With $x = 1$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n (-2)^{2n}}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n [(-2)^2]^n}{4^n} = \sum_{n=0}^{\infty} (-1)^n$, which diverges by the Divergence Test. With $x = 5$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{4^n} = \sum_{n=0}^{\infty} (-1)^n$, which diverges. Thus the interval of convergence is $(1, 5)$)

9) Find the exact sum of the series, if possible.

$$\sum_{n=2}^{\infty} \frac{2}{n^2 + 3n + 2}$$

(Rewrite the series using partial fractions, so that it becomes $\sum_{n=2}^{\infty} \left(\frac{2}{n+1} - \frac{2}{n+2}\right)$. Write out the first several terms of the series to find S_n:

$\sum_{n=2}^{\infty} \left(\frac{2}{n+1} - \frac{2}{n+2}\right) = \left(\frac{2}{3} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{2}{7}\right) + \cdots + \left(\frac{2}{n} - \frac{2}{n+1}\right) + \left(\frac{2}{n+1} - \frac{2}{n+2}\right) + \cdots$

Of the first n terms, all but the first and last cancel, so $S_n = \frac{2}{3} - \frac{2}{n+2}$, and $\lim_{n\to\infty} \left(\frac{2}{3} - \frac{2}{n+2}\right) = \frac{2}{3}$. Thus $\sum_{n=2}^{\infty} \frac{2}{n^2+3n+2} = \frac{2}{3}$.)

10) Find the first four nonzero terms in the Maclaurin series for $g(x) = \frac{\sin x}{e^x}$. (Solution posted on website)

11) Find a power series for the function, centered at c, and determine the interval of convergence.

$$f(x) = \frac{2}{3x-1}, \quad c = 1$$

(Rewrite: $\frac{2}{3x-1} = \frac{2}{3(x-1+1)-1} = \frac{2}{3(x-1)+2} = \frac{1}{1-\left[-\frac{3}{2}(x-1)\right]}$, so $a = 1$ and $r = -\frac{3}{2}(x-1)$ and so the power series is $\sum_{n=0}^{\infty}\left[-\frac{3}{2}(x-1)\right]^n$. For interval of convergence, we need $\left|-\frac{3}{2}(x-1)\right| < 1$, so $-1 < \frac{3}{2}(x-1) < 1 \rightarrow -\frac{2}{3} < x-1 < \frac{2}{3} \rightarrow \frac{1}{3} < x < \frac{5}{3}$, so interval is $\left(\frac{1}{3}, \frac{5}{3}\right)$.)

12) Find the first four nonzero terms of the Maclaurin series for the function $f(x) = \cos x \ln(1+x)$.

13) Use the definition of Taylor series to find the Taylor series, centered at $c = 1$, for the function $f(x) = \ln x$.

On the exam you will be expected to show all of your work. Note again that in the answers given here, not all work was shown!

Any problem similar to the ones shown here can appear on the exam.

You should also study all the homework assignments and handouts for this chapter, and go through the review exercises for chapter 9 on pp. 676-678 for more practice.