**Math 4329 MOCK FINAL** **Return in class on November 29, 2016**

Name: ______________________________________________

| Number | Max Possible | Points |
|---|---|---|
| 1 | 20 | |
| 2 | 18 | |
| 3 | 20 | |
| 4 | 18 | |
| 5 | 16 | |
| 6 | 20 | |
| 7 | 18 | |
| 8 | 20 | |
| Total | 150 | |
| **EXTRA CREDIT** | **50** | |

**1.** (20 pts) (**Quadrature Rules**)

(a) Find $c_1$ and $c_2$ in the following quadrature formula:

$$\int_0^1 f(x)dx \approx c_1 f(0) \ + \ c_2 f(1)$$

so that is exact for all polynomials of the largest degree possible. What is the degree of precision for this formula ?

(b) i. Approximate $\int_{-1}^{1} x^4 \, dx$ using the two point Gaussian quadrature rule with nodes $\pm 3^{-1/2}$ and weights 1.

ii. Calculate the exact integral $\int_{-1}^{1} x^4 \, dx$ and compare the error between the true value and the approximation obtained in (i).

**2.** (18 pts) (**Rootfinding techniques**)

(a) Consider the equation $e^{-x} = \sin(x)$. Find the interval $[a, b]$ containing the smallest positive root $\alpha$. Estimate the number $n$ of midpoints $c_n$ needed to obtain an approximate root that is accurate within an error tolerance of $10^{-9}$.
You may use the formula $|\alpha - c_n| \leq \frac{b-a}{2^n}$.

(b) Consider the fixed point iteration

$$x_{n+1} = 2 - (1 + c)x_n + c\, x_n^3.$$

Find the values of $c$ to ensure the convergence of the iterations generated by the above formula to $\alpha$ provided $x_0$ is chosen sufficiently close to the actual root $\alpha = 1$.

**3.** (20 pts) Gaussian Elimination

(a) Use Gaussian elimination with back substitution to solve the system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 3x_1 + 4x_2 + x_3 &= 6 \\ 2x_1 + 5x_2 + 7x_3 &= 2 \end{aligned}$$

Please specify the multipliers $m_{21}$, $m_{31}$ and $m_{32}$.

(b) Use the multipliers from the previous part (b) to form the LU factorization of the coefficient matrix of the linear system.

**4.** (18 pts) Consider the data $\{(-1, 2), (0, 1), (1, 1)\}$.

(a) Use Newton's divided difference formula to find the quadratic polynomial $p_2(x)$ that interpolates the above data. **Find the expression in the simplest form.**

(b) Use Lagrange's formula to find $p_2(x)$ and show that you got the same result as in (a).

**5.** (16 pts) Consider the roots of the quadratic equation $x^2 + bx + 1 = 0$ which can be expressed as

$$x_1 = \frac{-b + \sqrt{b^2 - 4}}{2}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4}}{2}.$$

(a) Assume that $b^2$ is much larger than 4, which root will suffer from the loss-of-signifincance errors and why ?

(b) Rewrite the expression for that specific root to avoid the loss-of-significance error.

**6.** (15 pts) Consider the following table:

| $x$ | $f(x)$ |
|---|---|
| 0.3 | 7.3891 |
| 0.4 | 7.4633 |
| 0.5 | 7.5383 |
| 0.6 | 7.6141 |
| 0.7 | 7.6906 |

where $x_{i+1} = x_i + h, \quad i = 0, 1, ..., 3..$

(a) Approximate $f'(0.5)$ using $D_h^+ f(0.5)$ and $h = 0.1$.

(b) Compute $D_h^{(2)} f(0.5)$ using the Central Difference Formula and step size $h = 0.2$.
Note: You may use the following formula for Central Difference Formula:

$$D_h^{(2)} f(x_1) = \frac{D_h^+ f(x_1) - D_h^- f(x_1)}{h}$$

(c) Compare the answer from (b) with the following approximation :

$$D_h^{(2)} f(x_1) = \frac{f(x_2) - 2f(x_1) + f(x_0)}{h^2},$$

with $x_1 = 0.5, \ h = 0.2.$

**7.** (10 pts) This question is related to floating-point numbers.

(a) Determine the number $x$ that has the following binary format:

$$(1111\ 1111\ 101)_2$$

(b) Furthermore, recall the **double** precision representation for any number y is

$$y = \sigma \cdot (1.a_1 a_2 a_3 \cdots a_{52}) \cdot 2^{E-1023}, \text{ where } E = (c_1 c_2 c_3 \cdots c_{11})_2.$$

Please express the number $x$ obtained above in its double precision representation.

**8.** (20 pts) Consider the Jacobi and Gauss Seidel methods applied to solve the following system:

$$\begin{aligned} 4x_1 + 3x_2 &= 7 \\ x_1 + 3x_2 &= 4 \end{aligned}$$

Compute $\mathbf{x}_J^{(k)}$, $\mathbf{x}_{GS}^{(k)}$ for $k = 1, 2$ with initial guess $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Do we have convergence ?

## EXTRA CREDIT

TBA