Math 4329 Mock Midterm 03 Review

1. (20 pts) State whether the following statement is true or false:

(a) Consider the following linear system:

$$\begin{aligned} x + y &= 0 \\ x + \frac{401}{400}y &= 1. \end{aligned}$$

The solution computed using Gaussian Elimination without pivoting and on a computer with three digits of significance is $x = -400,\ y = 400$.

(b) Consider the following iteration method described by

$$x^{(k+1)} = b + \begin{bmatrix} -4c & c \\ c & 4c \end{bmatrix} x^{(k)}, \quad k = 0, 1, \cdots$$

where c is a real constant. This iterative method diverges for values of $|c| \geq 0.2$.

2. (30 pts) Gaussian Elimination

(a) Use Gaussian elimination with back substitution to solve the system:

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 2x_1 + 6x_2 + 8x_3 &= 3 \\ 6x_1 + 8x_2 + 18x_3 &= 5 \end{aligned}$$

Please specify the multipliers m_{21}, m_{31} and m_{32}.

(b) Use the multipliers from the previous part (a) to form the LU factorization of the coefficient matrix of the linear system.

3. (10 pts) Consider the linear system:

$$\begin{aligned} 39x_1 + 40x_2 &= b_1 \\ 40x_1 + 41x_2 &= b_2 \end{aligned}$$

Calculate the condition number of the coefficient matrix. Is the system well-conditioned with respect to perturbations of the right hand side constants $\{b_1, b_2\}$?

4. (20 pts) Consider the Jacobi and Gauss Seidel methods applied to solve the following system:

$$\begin{aligned} 3x_1 - x_2 &= -4, \\ 2x_1 + 5x_2 &= 2. \end{aligned}$$

Compute $\mathbf{x}_J^{(k)}$, $\mathbf{x}_{GS}^{(k)}$ for $k = 1, 2$ with initial guess $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Do we have convergence for both the iterative methods?

5. (20 pts) This problem is a multiple choice problem do either **Option I** or **Option II**.

Option I: Consider the task of solving following linear system:

$$\begin{aligned} 0.729x_1 + 0.81x_2 + 0.9x_3 &= 0.6867 \\ x_1 + x_2 + x_3 &= 0.8338 \\ 1.331x_1 + 1.21x_2 + 1.1x_3 &= 1 \end{aligned}$$

on a computer using four-digit decimal machine with rounding. The true solution of this system is

$$x_1 = 0.2245 \quad x_2 = 0.2814 \quad x_3 = 0.3279,$$

correctly rounded to four digits.

If we apply Gaussian elimination (without pivoting) using four-digit decimal machine with rounding, we obtain the following computed solution

$$\hat{x}_1 = 0.2251, \quad \hat{x}_2 = 0.2790 \quad \hat{x}_3 = 0.3295.$$

Show that by applying one step of the residual correction method this solution can be corrected with the correction term $\hat{e}$ approximately given by

$$\hat{e} = [-0.0004471, 0.00215, -0.001504]$$

using eight-digit floating point decimal arithmetic and rounding.

Option II: Consider the problem of quadratic polynomial interpolation:

$$p(2) = 4, \quad p(-1) = 1, \quad p(1) = 0,$$

where $p(x)$ is a polynomial of degree at most 2.

(a) Transform the above polynomial interpolation problem to another problem of finding the solution of a system of linear equations which can be expressed in the form

$$Ax = b.$$

Clearly write down the matrix A and the vectors b and x.

(b) Show that the solution to the linear system obtained in part (a) can be obtained by solving two smaller linear systems:

$$Ly = b, \quad Ux = y,$$

where L and U are lower and upper triangular matrices respectively such that $A = LU$. Please explicitly write down the matrices L, U and the vector y.

For **Option I**, you may use the following residual correction method algorithm:

Input: $x^0 = \hat{x}$ obtained from using Gauss Elimination to solve $Ax = b$.
Tolerance $\varepsilon > 0$
Let $r^0 = b - Ax^0$
Solve for e^0 satisfying $Ae^0 = r^0$. **while** $|e^n| > \varepsilon$ **do**

> $x^{n+1} = x^n + e^n$
> Let $r^{n+1} = b - Ax^{n+1}$.
> Solve for e^{n+1} satisfying $Ae^{n+1} = r^{n+1}$.

end

Algorithm 1: Residual Correction Method