Math 4329 Practice Exam

The problems below are intended to provide students with some practice. Please note that this is not an exhaustive list. The scope of the exam includes all the material covered in the lectures and assignments.

1. This problem is based on Rootfinding techniques.

(a) Consider the equation $x\ e^x = \cos(x)$. Find the interval $[a, b]$ containing a **negative** root α.
Estimate the number n of midpoints c_n needed to obtain an approximate root that is accurate within an error tolerance of 10^{-9}.
You may use the formula $|\alpha - c_n| \le \frac{b-a}{2^n}$.

(b) Consider the fixed point iteration

$$x_{n+1} = 3 - (2 + c)x_n + c\ x_n^3.$$

Find the values of c to ensure the convergence of the iterations generated by the above formula to α provided x_0 is chosen sufficiently close to the actual root $\alpha = 1$.

2. This question is related to floating-point numbers.

(a) Determine the number x that has the following binary format:

$$(10011\ 1111\ 1001)_2$$

(b) Furthermore, recall the **double** precision representation for any number y is

$$y = \sigma \cdot (1.a_1a_2a_3 \cdots a_{52}) \cdot 2^{E-1023}, \text{ where } E = (c_1c_2c_3 \cdots c_{11})_2.$$

Please express the number x obtained above in its double precision representation.

3. (10 pts) How large should the degree $2n + 1$ be chosen in the Taylor expansion

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + (-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!} \sin(c)$$

to have

$$|\sin(x) - p_{2n+1}(x)| \le 0.01$$

for all $-2\pi \le x \le \pi$?
Note: p_{2n+1} denotes the Taylor polynomial of degree $2n+1$ for $f(x)$ about 0 and c denotes a real number between 0 and x.

4. This question is concerned with the loss-of-significance errors.
For each of the following functions $f(x)$ evaluated at $x = (0.01)^n$, $n = 1, 2 \cdots 10$, there is a loss of significance that occurs.

$$\text{(a) } f(x) = \frac{\sqrt{4 + x} - 2}{x},$$

$$\text{(b) } f(x) = \frac{x - \sin x}{x^3}.$$

Discuss how to avoid the loss of significance in the following calculations by reformulating $f(x)$ as a mathematically equivalent function $g(x)$. Please specify the function $g(x)$.

5. Consider the task of finding a root $\alpha \approx 1.2564$ of the following equation

$$f(x) := e^x - 2x - 1 = 0, \quad x \in [1, 2]. \tag{1}$$

We consider the following three fixed point iterative methods **Iter_1–Iter_3** to solve (1):

Iter_1: $x_{n+1} = \frac{e^{x_n}-1}{2}$

Iter_2: $x_{n+1} = e^{x_n} - x_n - 1$

Iter_3: $x_{n+1} = \ln(2x_n + 1)$.

Each of the iteration formulas **Iter_1–Iter_3** have the form

$$\boxed{x_{n+1} = g(x_n)}$$

for appropriately chosen continuous functions $g(x)$.

(a) Determine which of the fixed point iterations **Iter_1–Iter_3** will converge to the root α (provided the initial guess x_0 is chosen to be sufficiently close to α).

Furthermore, show that the fixed point iterative methods which converge, do so at a linear rate.

(b) Design a fixed point iterative method which converges **quadratically** (provided the initial guess x_0 is chosen sufficiently close to α) and assumes the following form:

$$\boxed{\textbf{Iter\_4:} \quad x_{n+1} = g_4(x_n)}$$

for a suitable choice of $g_4(x)$.

Please specify the function $g_4(x)$ characterizing the iterative method and provide sufficient reason for the quadratic convergence of **Iter_4**.

6. State whether the following statements are true or false:

I. If the Newton's method is used on $f(x) = 3x^3 + 2x + 1$ starting with $x_0 = 0$, the value of the next iterate $x_1 = 1/2$.

II. The iteration formula for the **Secant Method** applied to the equation $x^2 - 5 = 0$ to find the root α can be expressed in the following form:

$$x_{n+1} = \frac{x_{n-1}x_n + 5}{x_n + x_{n-1}}.$$

III. The quadratic polynomial $p_2(x)$ that interpolates the points $\{(1,3),(2,1),(3,2)\}$ is

$$p_2(x) = \frac{3}{2}x^2 - \frac{13}{2}x + 8.$$