Math 4329 FINAL REVIEW

This review is just a guide to supplement the problems assigned in the worksheets and discussed in the class.

1. Consider the linear system given below:

$$2x_1 - 3x_2 = -7 \quad (1)$$
$$x_1 + 3x_2 - 10x_3 = 9 \quad (2)$$
$$3x_1 + x_3 = 13 \quad (3)$$

Interchange the rows of the system of linear equations (1)–(3) to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply one step of the Gauss-Seidel method to approximate the solution to two significant digits. Assume an initial approximation of $x_1 = x_2 = x_3 = 0$.

2. How large should the degree $2n$ be chosen in the Taylor expansion

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + +(-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!} \cos(c)$$

to have

$$|\cos(x) - p_{2n}(x)| \leq 0.01$$

for all $0 \leq x \leq \pi$?
Note p_{2n} denotes the Taylor polynomial of degree $2n$ for $f(x)$ about 0 and c denotes a real number between 0 and x.

3. Use the first order Taylor polynomial with the remainder term to avoid the loss-of-significance errors in the following formula when x is near 0:

$$\frac{e^x - e^{-x}}{2x}.$$

4. Let $f(x) = \frac{1}{3x+1}$, $x_0 = 0, x_1 = 2, x_2 = 3$.

(a) Calculate the piecewise linear interpolation polynomial.

(b) Calculate $f[x_0, x_1]$, $f[x_0, x_1, x_2]$ and the quadratic interpolation polynomial $p_2(x)$.

5. For the integral

$$I = \int_0^1 \sqrt{x} e^x dx,$$

calculate $I - T_4$ and $I - S_4$ where T_4 and S_4 are the Trapezoidal and Simpson quadrature rules (the Trapezoidal and Simpson quadrature formulas will be provided in the exam).

6. For the linear system $\mathbf{Ax} = \mathbf{b}$, consider the following iterative scheme

$$\mathbf{x_{n+1}} = \mathbf{b} + \mathbf{Mx_n} \quad \mathbf{n} = \mathbf{0}, \mathbf{1}, \cdots \tag{4}$$

where $\mathbf{M} := \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, where α and β are constants.

Under what conditions on α and β will the iterative scheme converge for a given initial guess?

7. Determine constants a, b, c that will produce a quadrature formula

$$\int_{-1}^{1} f(x)\, dx = af(-1) + bf(1) + cf'(0.5),$$

that has degree of precision as high as possible.

8. Consider the Jacobi and Gauss Seidel methods applied to solve the following system:

$$\begin{aligned} 3x_1 - x_2 &= -4, \\ 2x_1 + 5x_2 &= 2. \end{aligned}$$

Compute $\mathbf{x}_J^{(k)}$, $\mathbf{x}_{GS}^{(k)}$ for $k = 1, 2$ with initial guess $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Do we have convergence?

9. Check whether the following function is a spline:

$$f(x) = \begin{cases} x^3 & -1 \le x \le 0, \\ 3x^3 - x^2 & 0 < x < 1, \\ 2x & 1 \le x \le 2. \end{cases}$$

10. (**Rootfinding techniques**)

(a) Consider the equation $x\, e^x = \cos(x)$. Find the interval $[a, b]$ containing the smallest positive root α. Estimate the number n of midpoints c_n needed to obtain an approximate root that is accurate within an error tolerance of 10^{-9}.
You may use the formula $|\alpha - c_n| \le \frac{b-a}{2^n}$.

(b) Consider the fixed point iteration

$$x_{n+1} = 3 - (2 + c)x_n + c\, x_n^3.$$

Find the values of c to ensure the convergence of the iterations generated by the above formula to α provided x_0 is chosen sufficiently close to the actual root $\alpha = 1$.

11. Gaussian Elimination

(a) Use Gaussian elimination with back substitution to solve the system:

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 2x_1 + 6x_2 + 8x_3 &= 3 \\ 6x_1 + 8x_2 + 18x_3 &= 5 \end{aligned}$$

Please specify the multipliers m_{21}, m_{31} and m_{32}.

(b) Use the multipliers from the previous part to form the LU factorization of the coefficient matrix of the linear system.

12. This question is related to floating-point numbers.

(a) Determine the number x that has the following binary format:

$$(1111\ 1111\ 101)_2$$

(b) Furthermore, recall the **double** precision representation for any number y is

$$y = \sigma \cdot (1.a_1a_2a_3 \cdots a_{52}) \cdot 2^{E-1023}, \text{ where } E = (c_1c_2c_3 \cdots c_{11})_2.$$

Please express the number x obtained above in its double precision representation.

13. State whether the following statements are true or false:

(a) The following function is a spline:

$$f(x) = \begin{cases} x^3 & -1 \le x \le 0, \\ 3x^3 - x^2 & 0 < x < 1, \\ 2x & 1 \le x \le 2. \end{cases}$$

(b) If the Newton's method is used on $f(x) = 3x^3 + 2x + 1$ starting with $x_0 = 0$, the value of the next iterate $x_1 = 1/2$.

(c) Consider the following linear system:

$$\begin{aligned} x + y &= 0 \\ x + \frac{801}{800}y &= 1. \end{aligned}$$

The solution computed using Gaussian Elimination on a computer with four digits of significance is $x = -800,\ y = 800$.

(d) The Taylor Polynomial of degree 3 approximates $\cos(x)$ function on $(-\pi/4, \pi/4)$ with an error no greater than 10^{-3}.

(e) Consider the roots of the equation (in x^{-1}) of

$$x^{-2} + bx^{-1} + 1 = 0, \quad \text{with} \quad b > 0,$$

which can be expressed as:

$$x_1 = \frac{2}{-b + \sqrt{b^2 - 4}}, \quad x_2 = \frac{2}{-b - \sqrt{b^2 - 4}}.$$

Assume that b^2 is much larger than 4, x_2 will suffer from the loss-of-significance error.

14. Consider the following table:

x	$f(x)$
0.3	7.3891
0.4	7.4633
0.5	7.5383
0.6	7.6141
0.7	7.6906

where $x_{i+1} = x_i + h, \quad i = 0, 1, ..., 3..$

(a) Approximate $f'(0.5)$ using $D_h^+ f(0.5)$ and $h = 0.1$.

(b) Compute $D_h^{(2)} f(0.5)$ using the Central Difference Formula and step size $h = 0.2$.
Note: You may use the following formula for Central Difference Formula:

$$D_h^{(2)} f(x_1) = \frac{D_h^+ f(x_1) - D_h^- f(x_1)}{h}.$$

(c) Compare the answer from (b) with the following approximation :

$$D_h^{(2)} f(x_1) = \frac{f(x_2) - 2f(x_1) + f(x_0)}{h^2},$$

with $x_1 = 0.5, \ h = 0.2$.

15. (a) Approximate $\int_{-1}^{1} x^8 \, dx$ using the two point Gaussian quadrature rule with nodes $\pm 3^{-1/2}$ and weights 1.

(b) Calculate the exact integral $\int_{-1}^{1} x^8 \, dx$ and compare the error between the true value and the approximation obtained in the previous part.