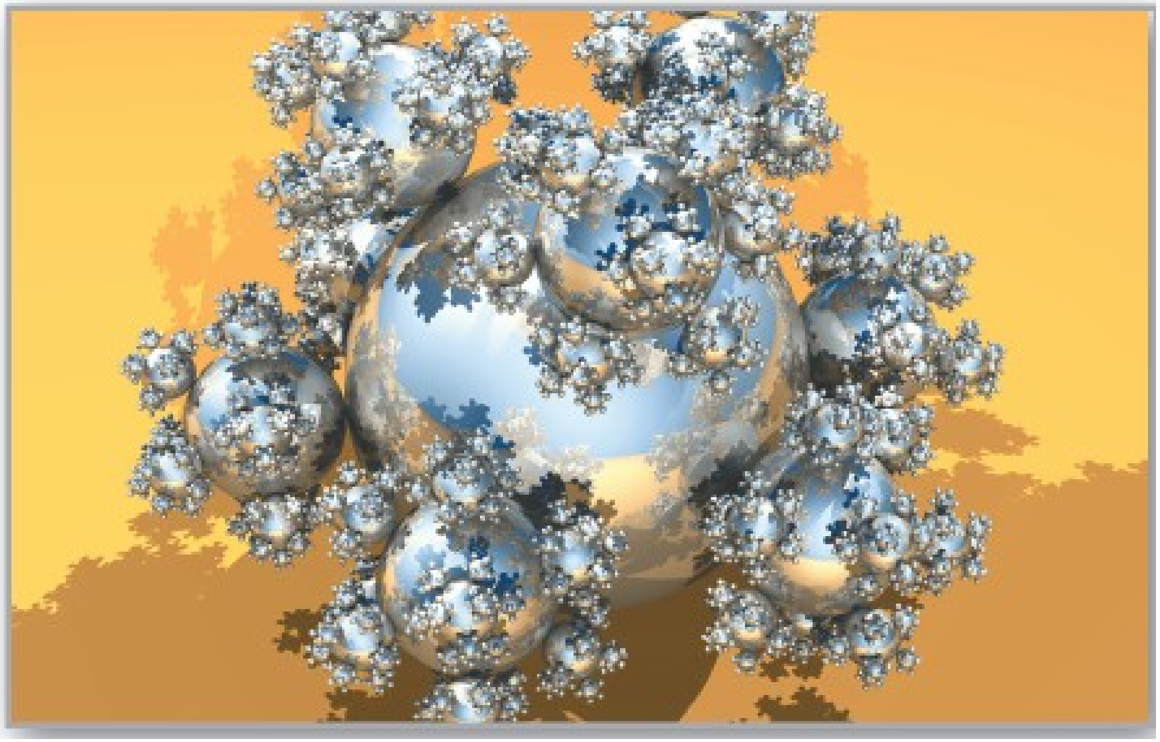
9 Infinite Series



9.9 Representation of Functions by Power Series

Objectives

- Find a geometric power series that represents a function.
- Construct a power series using series operations.

Geometric Power Series

Geometric Power Series

Consider the function given by $f(x) = 1/(1 - x)$. The form of f closely resembles the sum of a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1.$$

In other words, when $a = 1$ and $r = x$, a power series representation for $1/(1 - x)$, centered at 0, is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} ar^n$$

$$= \sum_{n=0}^{\infty} x^n$$

$$= 1 + x + x^2 + x^3 + \cdot \cdot \cdot, \quad |x| < 1.$$

Geometric Power Series

Of course, this series represents $f(x) = 1/(1 - x)$ only on the interval $(-1, 1)$, whereas f is defined for all $x \neq 1$, as shown in Figure 9.22.

To represent f in another interval, you must develop a different series.

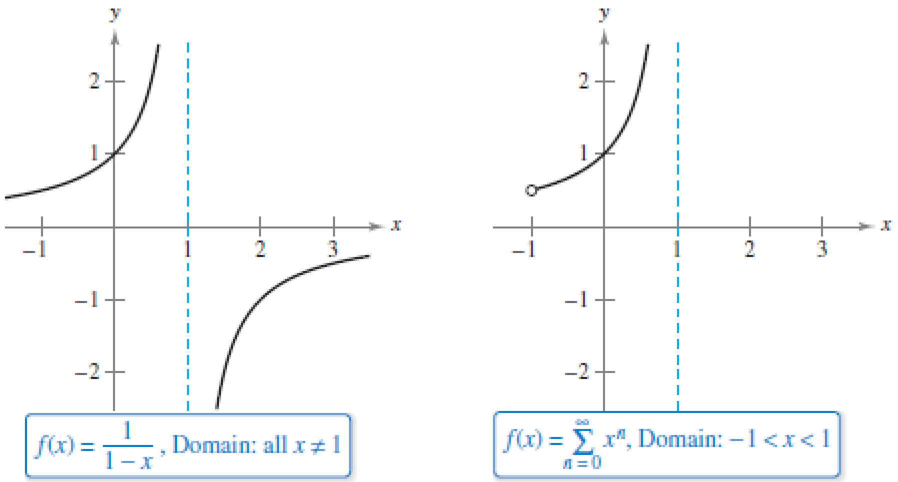


Figure 9.22

Geometric Power Series

For instance, to obtain the power series centered at –1, you could write

$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{1/2}{1-[(x+1)/2]} = \frac{a}{1-r}$$

which implies that $a = \frac{1}{2}$ and $r = (x+1)/2.$

So, for $|x+1| < 2,$ you have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{x+1}{2}\right)^n$$

$$= \frac{1}{2}\left[1 + \frac{(x+1)}{2} + \frac{(x+1)^2}{4} + \frac{(x+1)^3}{8} + \cdots\right], \quad |x+1| < 2$$

which converges on the interval (–3, 1).

Example 1 – *Finding a Geometric Power Series Centered at* 0

Find a power series for $f(x) = \dfrac{4}{x+2}$, centered at 0.

Solution:

Writing $f(x)$ in the form $a/(1 - r)$ produces

$$\frac{4}{2+x} = \frac{2}{1-(-x/2)} = \frac{a}{1-r}$$

which implies that $a = 2$ and $r = -x/2$.

Example 1 – *Solution*

cont'd

So, the power series for $f(x)$ is

$$\frac{4}{x+2} = \sum_{n=0}^{\infty} ar^n$$

$$= \sum_{n=0}^{\infty} 2\left(-\frac{x}{2}\right)^n$$

$$= 2\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdot \cdot \cdot\right).$$

This power series converges when

$$\left|-\frac{x}{2}\right| < 1$$

which implies that the interval of convergence is (–2, 2).

Operations with Power Series

Operations with Power Series

Operations with Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

Example 3 – *Adding Two Power Series*

Find a power series for

$$f(x) = \frac{3x - 1}{x^2 - 1}$$

centered at 0.

Solution:

Using partial fractions, you can write $f(x)$ as

$$\frac{3x - 1}{x^2 - 1} = \frac{2}{x + 1} + \frac{1}{x - 1}.$$

Example 3 – *Solution*

cont'd

By adding the two geometric power series

$$\frac{2}{x+1} = \frac{2}{1-(-x)} = \sum_{n=0}^{\infty} 2(-1)^n x^n, \quad |x| < 1$$

and

$$\frac{1}{x-1} = \frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

you obtain the following power series shown below.

$$\frac{3x-1}{x^2-1} = \sum_{n=0}^{\infty} [2(-1)^n - 1]x^n = 1 - 3x + x^2 - 3x^3 + x^4 - \cdots$$

The interval of convergence for this power series is (–1, 1).