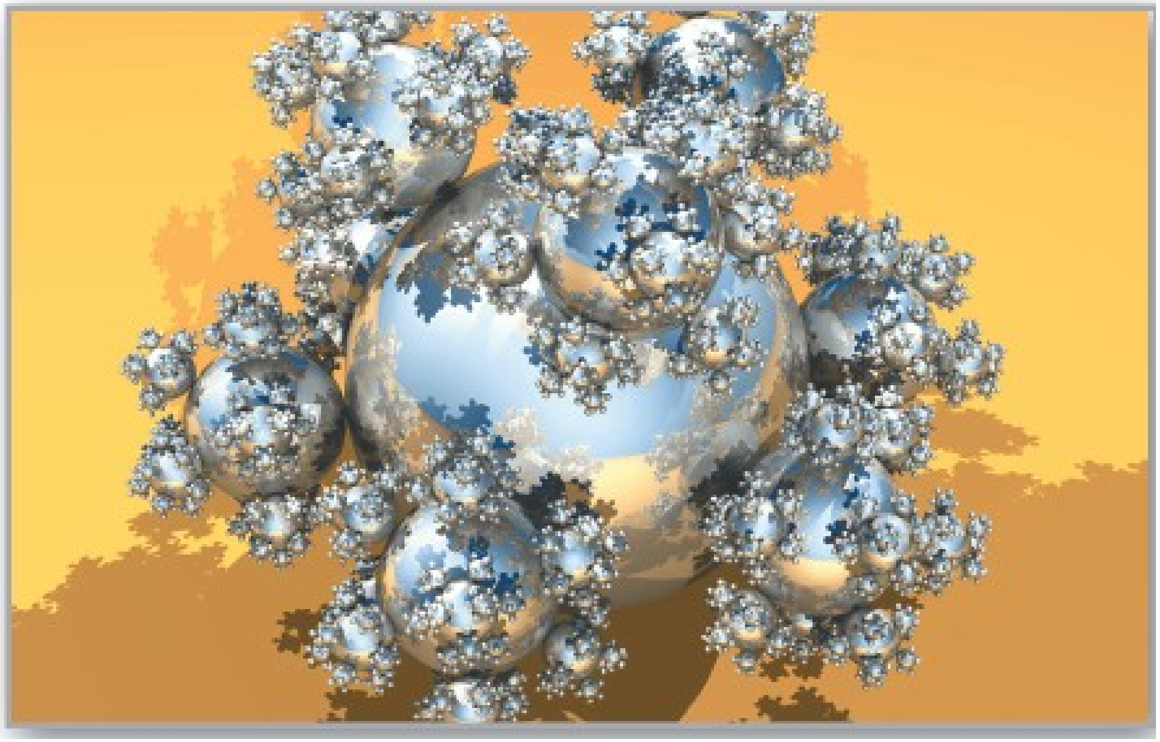
9 Infinite Series



9.8 Power Series

Objectives

- Understand the definition of a power series.
- Find the radius and interval of convergence of a power series.
- Determine the endpoint convergence of a power series.
- Differentiate and integrate a power series.

Power Series

Power Series

An important function $f(x) = e^x$ can be represented *exactly* by an infinite series called a **power series.** For example, the power series representation for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

For each real number *x*, it can be shown that the infinite series on the right converges to the number e^x.

Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series.** More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a **power series centered at** c, where c is a constant.

Example 1 – *Power Series*

a. The following power series is centered at 0.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

b. The following power series is centered at –1.

$$\sum_{n=0}^{\infty} (-1)^n (x + 1)^n = 1 - (x + 1) + (x + 1)^2 - (x + 1)^3 + \cdots$$

c. The following power series is centered at 1.

$$\sum_{n=1}^{\infty} \frac{1}{n} (x - 1)^n = (x - 1) + \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 + \cdots$$

Radius and Interval of Convergence

Radius and Interval of Convergence

A power series in *x* can be viewed as a function of *x*

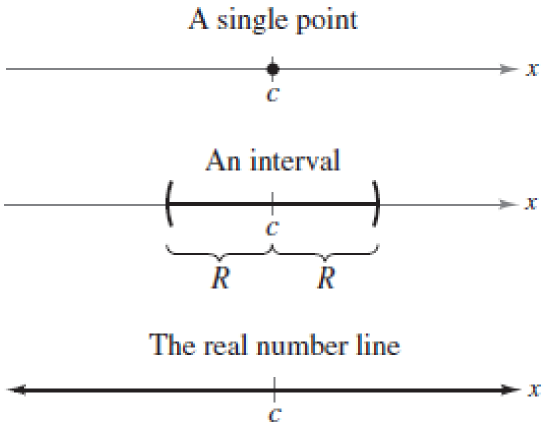
$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

where the *domain of f* is the set of all *x* for which the power series converges. Of course, every power series converges at its center *c* because

$$\begin{aligned} f(c) &= \sum_{n=0}^{\infty} a_n(c - c)^n \\ &= a_0(1) + 0 + 0 + \cdots + 0 + \cdots \\ &= a_0. \end{aligned}$$

Radius and Interval of Convergence

So, *c* always lies in the domain of *f*. Theorem 9.20 (to follow) states that the domain of a power series can take three basic forms: a single point, an interval centered at *c*, or the entire real number line, as shown in Figure 9.17.



The domain of a power series has only three basic forms: a single point, an interval centered at *c*, or the entire real number line.

Figure 9.17

Radius and Interval of Convergence

THEOREM 9.20 Convergence of a Power Series

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for

$$|x - c| < R$$

and diverges for

$$|x - c| > R.$$

3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, then the radius of convergence is $R = 0$. If the series converges for all x, then the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

Example 2 – *Finding the Radius of Convergence*

Find the radius of convergence of $\sum_{n=0}^{\infty} n!x^n$.

Solution:

For $x = 0$, you obtain

$$f(0) = \sum_{n=0}^{\infty} n!0^n = 1 + 0 + 0 + \cdots = 1.$$

For any fixed value of x such that $|x| > 0$, let $u_n = n!x^n$.

Then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$

$$= |x| \lim_{n\to\infty} (n+1)$$

$$= \infty.$$

Example 2 – *Solution*

cont'd

Therefore, by the Ratio Test, the series diverges for $|x| > 0$ and converges only at its center, 0.

So, the radius of convergence is $R = 0$.

Endpoint Convergence

Endpoint Convergence

For a power series whose radius of convergence is a finite number R, Theorem 9.20 says nothing about the convergence at the *endpoints* of the interval of convergence.

Each endpoint must be tested separately for convergence or divergence.

Endpoint Convergence

As a result, the interval of convergence of a power series can take any one of the six forms shown in Figure 9.18.

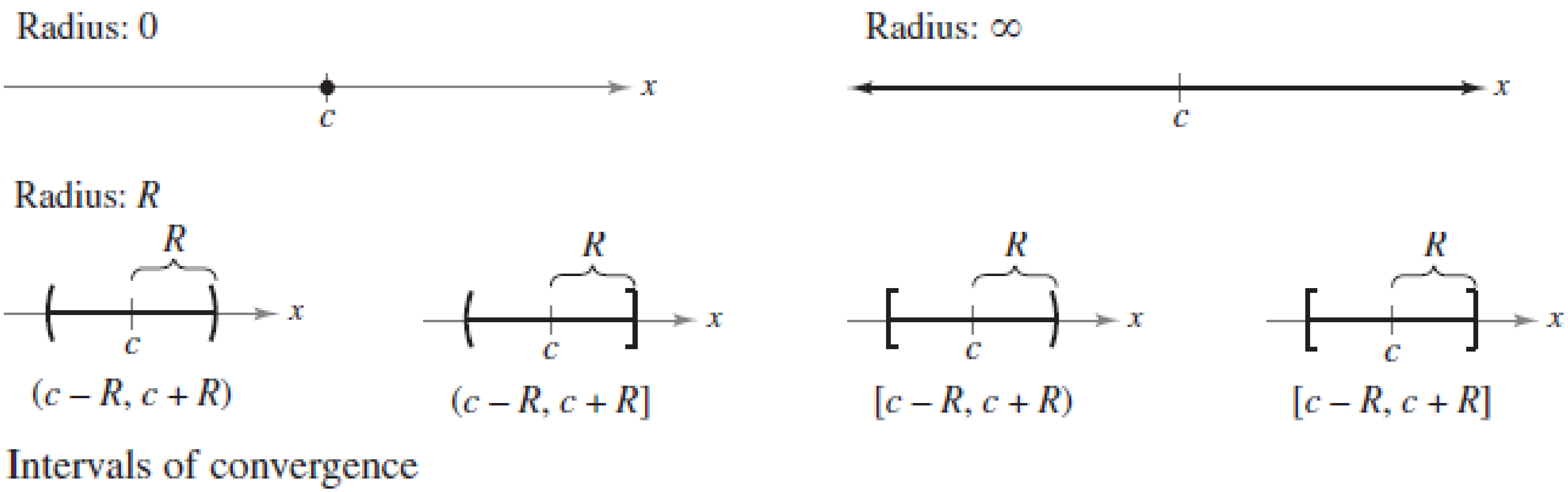


Figure 9.18

Example 5 – *Finding the Interval of Convergence*

Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Solution:

Letting $u_n = x^n/n$ produces

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{\dfrac{x^{n+1}}{(n+1)}}{\dfrac{x^n}{n}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{nx}{n+1}\right|$$

$$= |x|.$$

Example 5 – *Solution*

cont'd

So, by the Ratio Test, the radius of convergence is $R = 1$.

Moreover, because the series is centered at 0, it converges in the interval $(–1, 1)$.

This interval, however, is not necessarily the *interval of convergence*.

To determine this, you must test for convergence at each endpoint.

When $x = 1$, you obtain the *divergent* harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots . \qquad \text{Diverges when } x = 1$$

Example 5 – *Solution*

cont'd

When $x = -1$, you obtain the *convergent* alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots.$$

Converges when $x = -1$

So, the interval of convergence for the series is [–1, 1), as shown in Figure 9.19.

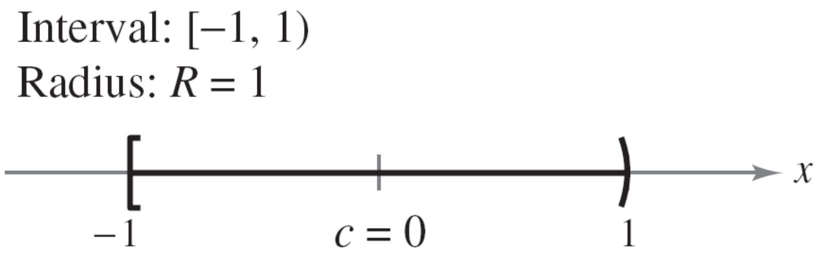


Figure 9.19

Differentiation and Integration of Power Series

Differentiation and Integration of Power Series

THEOREM 9.21 Properties of Functions Defined by Power Series

If the function

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$
$$= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

has a radius of convergence of $R > 0$, then, on the interval

$$(c - R, c + R)$$

f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$
$$= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

2. $\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$
$$= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence*, however, may differ as a result of the behavior at the endpoints.

Example 8 – *Intervals of Convergence for f(x), f'(x), and ∫f(x)dx*

Consider the function given by

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots.$$

Find the interval of convergence for each of the following.

a. $\int f(x)\,dx$

b. $f(x)$

c. $f'(x)$

Example 8 – *Solution*

cont'd

By Theorem 9.21, you have

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1}$$

$$= 1 + x + x^2 + x^3 + \cdots$$

and

$$\int f(x)\,dx = C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$$

$$= C + \frac{x^2}{1 \cdot 2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} + \cdots.$$

By the Ratio Test, you can show that each series has a radius of convergence of $R = 1$.

Considering the interval (–1, 1) you have the following.

Example 8(a) – *Solution*

cont'd

For $\int f(x)dx$, the series

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$$

Interval of convergence: $[-1, 1]$

converges for $x = \pm 1$, and its interval of convergence is $[-1, 1]$. See Figure 9.21(a).

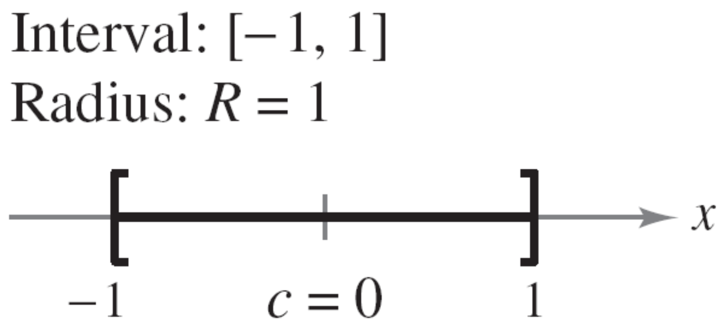


Figure 9.21(a)

Example 8(b) – *Solution*

cont'd

For *f*(*x*), the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Interval of convergence: $[-1, 1)$

converges for *x* = –1, and diverges for *x* = 1.

So, its interval of convergence is [–1, 1).

See Figure 9.21(b).

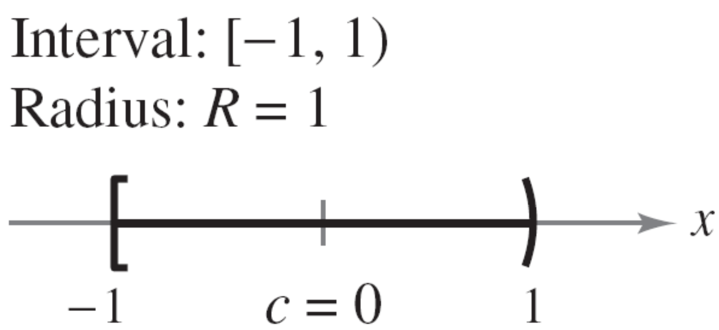


Figure 9.21(b)

Example 8(c) – *Solution*

cont'd

For $f'(x)$, the series

$$\sum_{n=1}^{\infty} x^{n-1} \qquad \text{Interval of convergence: } (-1, 1)$$

diverges for $x = \pm 1$, and its interval of convergence is $(-1, 1)$. See Figure 9.21(c).

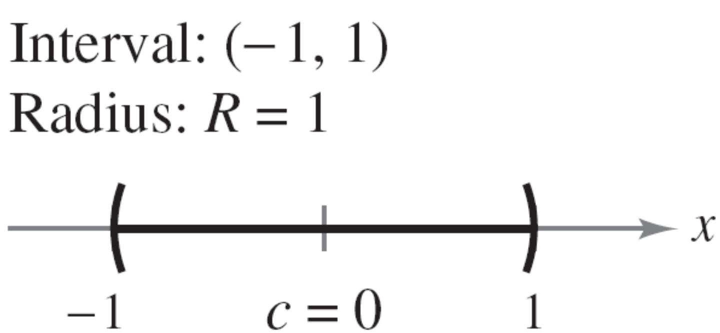


Figure 9.21(c)