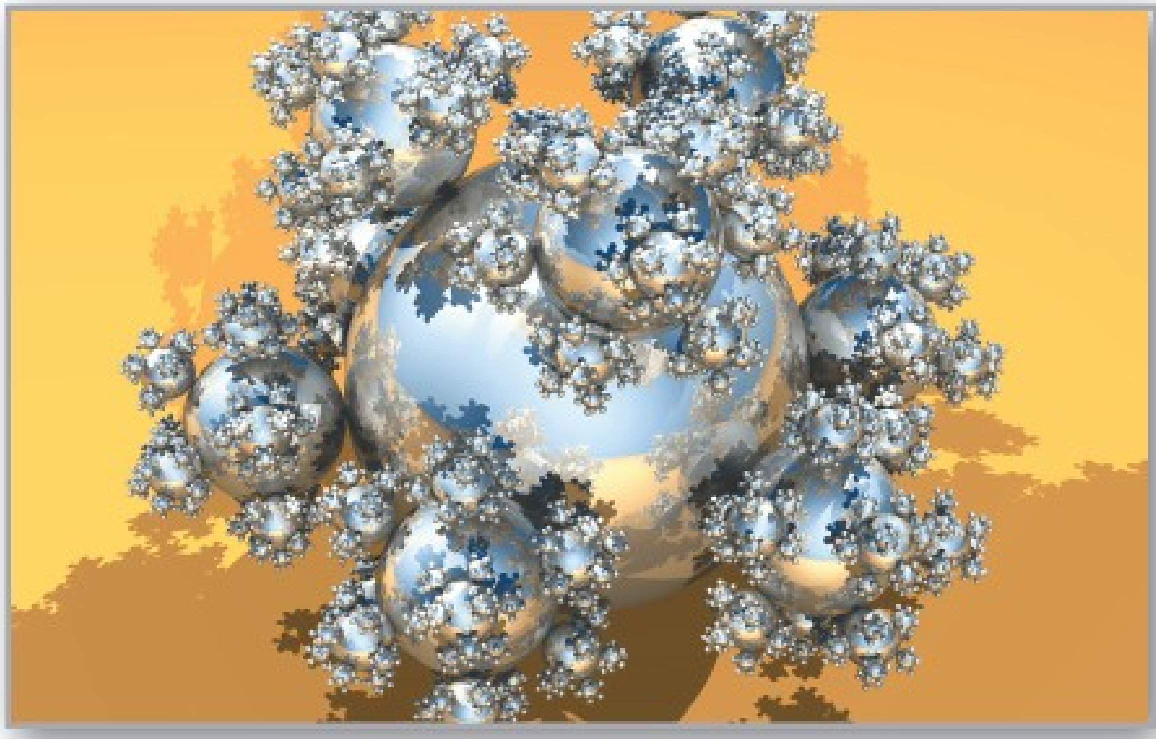
9 Infinite Series



9.4 Comparisons of Series

Objectives

- Use the Direct Comparison Test to determine whether a series converges or diverges.

- Use the Limit Comparison Test to determine whether a series converges or diverges.

Direct Comparison Test

Direct Comparison Test

For the convergence tests the terms of the series have to be fairly simple and the series must have special characteristics in order for the convergence tests to be applied.

A slight deviation from these special characteristics can make a test nonapplicable.

Direct Comparison Test

For example, in the following pairs, the second series cannot be tested by the same convergence test as the first series even though it is similar to the first.

1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is geometric, but $\sum_{n=0}^{\infty} \frac{n}{2^n}$ is not.

2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series, but $\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$ is not.

3. $a_n = \frac{n}{(n^2 + 3)^2}$ is easily integrated, but $b_n = \frac{n^2}{(n^2 + 3)^2}$ is not.

Direct Comparison Test

THEOREM 9.12 Direct Comparison Test

Let $0 < a_n \leq b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Example 1 – *Using the Direct Comparison Test*

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$.

Solution :

This series resembles

$$\sum_{n=1}^{\infty} \frac{1}{3^n}. \qquad \text{Convergent geometric series}$$

Term-by-term comparison yields

$$a_n = \frac{1}{2 + 3^n} < \frac{1}{3^n} = b_n, \quad n \geq 1.$$

So, by the Direct Comparison Test, the series converges.

Limit Comparison Test

Limit Comparison Test

Sometimes a series closely resembles a p -series or a geometric series, yet you cannot establish the term-by-term comparison necessary to apply the Direct Comparison Test. Under these circumstances you may be able to apply a second comparison test, called the **Limit Comparison Test.**

THEOREM 9.13 Limit Comparison Test

If $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where L is *finite and positive*, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.

Example 3 – *Using the Limit Comparison Test*

Show that the following general harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{an+b}, \quad a > 0, \quad b > 0$$

Solution:

By comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$ Divergent harmonic series

you have

$$\lim_{n\to\infty} \frac{1/(an+b)}{1/n} = \lim_{n\to\infty} \frac{n}{an+b} = \frac{1}{a}.$$

Because this limit is greater than 0, you can conclude from the Limit Comparison Test that the given series diverges.