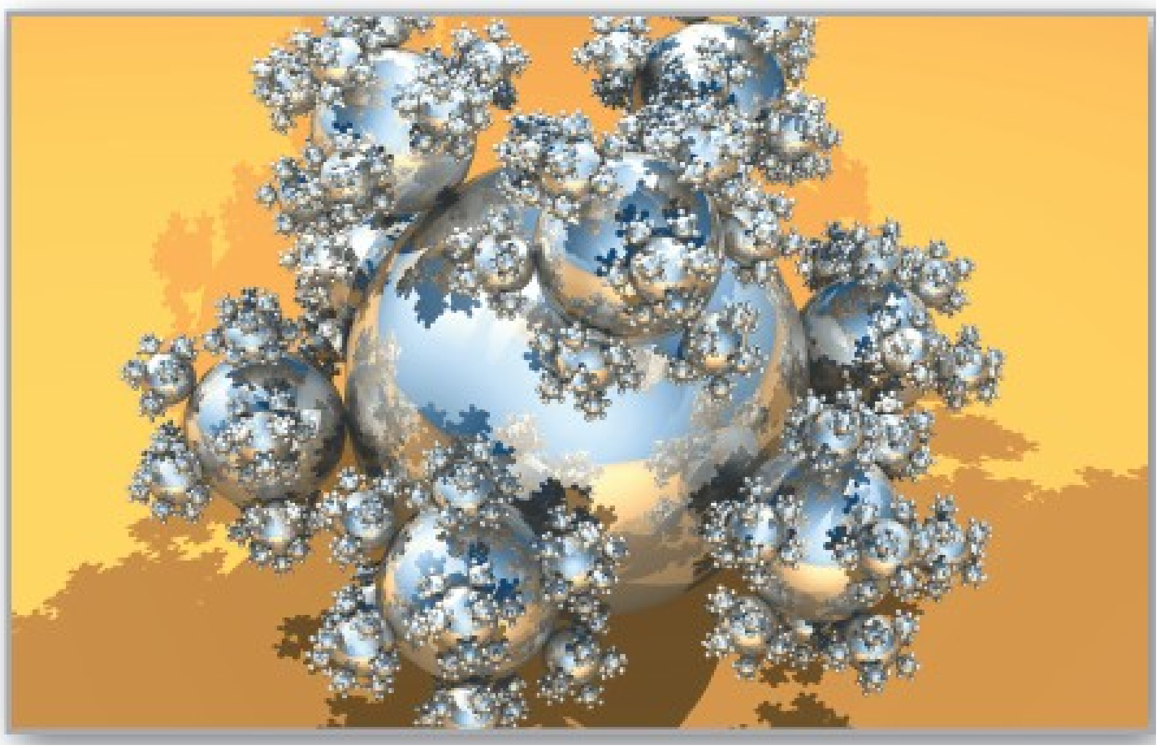
9 Infinite Series



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9.3 The Integral Test and *p*-Series

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Objectives

- Use the Integral Test to determine whether an infinite series converges or diverges.

- Use properties of *p*-series and harmonic series.

The Integral Test

The Integral Test

THEOREM 9.10 The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

Example 1 – *Using the Integral Test*

Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$.

Solution:

The function $f(x) = x/(x^2 + 1)$ is positive and continuous for $x \geq 1$.

To determine whether *f* is decreasing, find the derivative.

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{-x^2 + 1}{(x^2 + 1)^2}$$

Example 1 – *Solution*

cont'd

So, $f'(x) < 0$ for $x > 1$ and it follows that f satisfies the conditions for the Integral Test.

You can integrate to obtain

$$\begin{aligned}
\int_1^{\infty} \frac{x}{x^2+1}\,dx &= \frac{1}{2}\int_1^{\infty} \frac{2x}{x^2+1}\,dx \\
&= \frac{1}{2}\lim_{b\to\infty}\int_1^{b} \frac{2x}{x^2+1}\,dx \\
&= \frac{1}{2}\lim_{b\to\infty}\Big[\ln(x^2+1)\Big]_1^b \\
&= \frac{1}{2}\lim_{b\to\infty}\left[\ln(b^2+1) - \ln 2\right] \\
&= \infty.
\end{aligned}$$

So, the series *diverges*.

p-Series and Harmonic Series

p-Series and Harmonic Series

A second type of series has a simple arithmetic test for convergence or divergence. A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdot \cdot \cdot \qquad p\text{-series}$$

is a ***p*-series,** where p is a positive constant. For $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdot \cdot \cdot \qquad \text{Harmonic series}$$

is the **harmonic** series.

p-Series and Harmonic Series

A **general harmonic series** of the form $\Sigma 1/(an + b)$. In music, strings of the same material, diameter, and tension, whose lengths form a harmonic series, produce harmonic tones.

THEOREM 9.11 Convergence of *p*-Series

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdot \cdot \cdot$$

converges for $p > 1$, and diverges for $0 < p \leq 1$.

Example 3 – *Convergent and Divergent p-Series*

Discuss the convergence or divergence of (a) the harmonic series and (b) the *p*-series with *p* = 2.

Solution:

a. From Theorem 9.11, it follows that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \qquad p = 1$$

diverges.

b. From Theorem 9.11, it follows that the *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \qquad p = 2$$

converges.