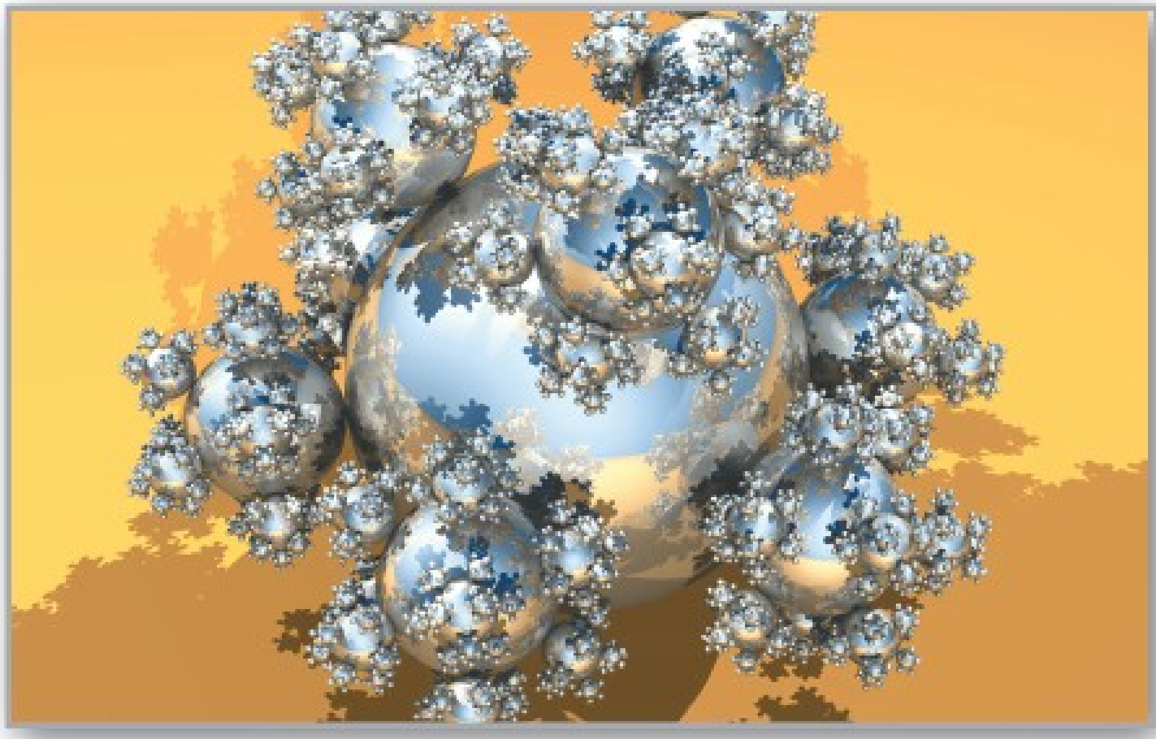
9 Infinite Series



9.1 Sequences

Objectives

- List the terms of a sequence.
- Determine whether a sequence converges or diverges.
- Write a formula for the *n*th term of a sequence.
- Use properties of monotonic sequences and bounded sequences.

Sequences

Sequences

A **sequence** is defined as a function whose domain is the set of positive integers. Although a sequence is a function, it is common to represent sequences by subscript notation rather than by the standard function notation. For instance, in the sequence

$$\begin{array}{ccccccccc} 1, & 2, & 3, & 4, & \ldots, & n, & \ldots & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \text{Sequence} \\ a_1, & a_2, & a_3, & a_4, & \ldots, & a_n, & \ldots & & \end{array}$$

1 is mapped onto a_1, 2 is mapped onto a_2, and so on. The numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the **terms** of the sequence. The number a_n is the ***n*th term** of the sequence, and the entire sequence is denoted by $\{a_n\}$.

Example 1 – *Listing the Terms of a Sequence*

a. The terms of the sequence $\{a_n\} = \{3 + (-1)^n\}$ are

$3 + (-1)^1,\ 3 + (-1)^2,\ 3 + (-1)^3,\ 3 + (-1)^4,\ \ldots$

$2, \quad 4, \quad 2, \quad 4, \quad \ldots.$

b. The terms of the sequence $\{b_n\} = \left\{\dfrac{n}{1-2n}\right\}$ are

$$\frac{1}{1-2\cdot 1},\ \frac{2}{1-2\cdot 2},\ \frac{3}{1-2\cdot 3},\ \frac{4}{1-2\cdot 4},\ \ldots$$

$$-1, \quad -\frac{2}{3}, \quad -\frac{3}{5}, \quad -\frac{4}{7}, \quad \ldots.$$

Example 1 – *Listing the Terms of a Sequence* cont'd

c. The terms of the sequence $\{c_n\} = \left\{\dfrac{n^2}{2^n - 1}\right\}$ are

$$\frac{1^2}{2^1 - 1}, \frac{2^2}{2^2 - 1}, \frac{3^2}{2^3 - 1}, \frac{4^2}{2^4 - 1}, \ldots$$

$$\frac{1}{1}, \quad \frac{4}{3}, \quad \frac{9}{7}, \quad \frac{16}{15}, \quad \ldots .$$

d. The terms of the **recursively defined** sequence $\{d_n\}$, where $d_1 = 25$ and $d_{n+1} = d_n - 5$, are

25, 25 – 5 = 20, 20 – 5 = 15, 15 – 5 = 10,. . . . .

Limit of a Sequence

Limit of a Sequence

Sequences whose terms approach limiting values are said to **converge.** For instance, the sequence $\{1/2^n\}$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

converges to 0, as indicated in the following definition.

Definition of the Limit of a Sequence

Let L be a real number. The **limit** of a sequence $\{a_n\}$ is L, written as

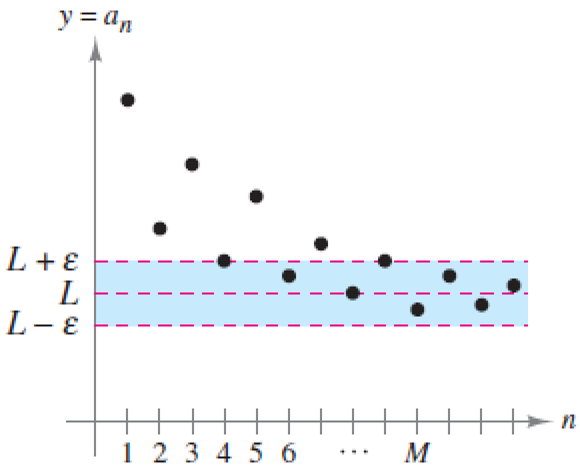
$$\lim_{n\to\infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence **converges** to L. If the limit of a sequence does not exist, then the sequence **diverges.**

Limit of a Sequence

Graphically, this definition says that eventually (for $n > M$ and $\varepsilon > 0$) the terms of a sequence that converges to L will lie within the band between the lines $y = L + \varepsilon$ and $y = L - \varepsilon$ as shown in Figure 9.1.

If a sequence $\{a_n\}$ agrees with a function f at every positive integer, and if $f(x)$ approaches a limit L as $x \to \infty$, the sequence must converge to the same limit L.



For $n > M$, the terms of the sequence all lie within ε units of L.

Figure 9.1

Limit of a Sequence

THEOREM 9.1 Limit of a Sequence

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x \to \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n \to \infty} a_n = L.$$

Example 2 – *Finding the Limit of a Sequence*

Find the limit of the sequence whose *n*th term is

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Solution:

You learned that $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e.$

So, you can apply Theorem 9.1 to conclude that

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Limit of a Sequence

There are different ways in which a sequence can fail to have a limit.

One way is that the terms of the sequence increase without bound or decrease without bound.

These cases are written symbolically, as shown below.

Terms increase without bound: $\lim_{n \to \infty} a_n = \infty$

Terms decrease without bound: $\lim_{n \to \infty} a_n = -\infty$

Limit of a Sequence

THEOREM 9.2 Properties of Limits of Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$

2. $\lim_{n\to\infty} ca_n = cL,\ c$ is any real number.

3. $\lim_{n\to\infty} (a_n b_n) = LK$

4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{K},\ b_n \neq 0$ and $K \neq 0$

Limit of a Sequence

The symbol $n!$ (read "n factorial") is used to simplify some of these formulas. Let n be a positive integer; then ***n* factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n - 1) \cdot n.$$

As a special case, **zero factorial** is defined as $0! = 1$. From this definition, you can see that $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, and so on.

Limit of a Sequence

Factorials follow the same conventions for order of operations as exponents. That is, just as $2x^3$ and $(2x)^3$

imply different order of operations, $2n!$ and $(2n)!$ imply the orders

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot n)$$

and

$$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot n \cdot (n + 1) \cdot \cdot \cdot 2n$$

respectively.

Limit of a Sequence

Another useful limit theorem that can be rewritten for sequence is the Squeeze Theorem.

THEOREM 9.3 Squeeze Theorem for Sequences

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then $\lim_{n\to\infty} c_n = L$.

Example 5 – *Using the Squeeze Theorem*

Show that the sequence $\{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$ converges, and find its limit.

Solution:

To apply the Squeeze Theorem, you must find two convergent sequences that can be related to the given sequence.

Two possibilities are $a_n = -1/2^n$ and $b_n = 1/2^n$, both of which converge to 0.

By comparing the term $n!$ with 2^n, you can see that,

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots n = 24 \cdot \underbrace{5 \cdot 6 \cdots n}_{n-4 \text{ factors}} \qquad (n \geq 4)$$

and

$$2^n = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdots 2 = 16 \cdot \underbrace{2 \cdot 2 \cdots 2}_{n-4 \text{ factors}}. \qquad (n \geq 4)$$

Example 5 – *Solution*

cont'd

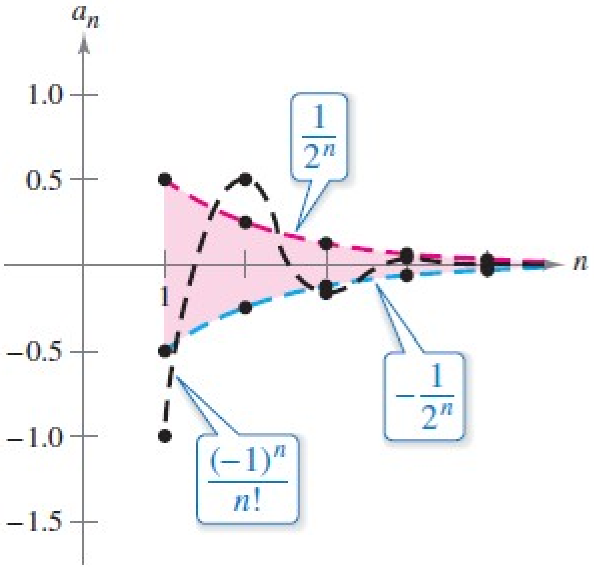
This implies that for $n \geq 4$, $2^n < n!$, and you have

$$\frac{-1}{2^n} \leq (-1)^n \frac{1}{n!} \leq \frac{1}{2^n}, \quad n \geq 4$$

as shown in Figure 9.2.

So, by the Squeeze Theorem it follows that

$$\lim_{n\to\infty} (-1)^n \frac{1}{n!} = 0.$$



For $n \geq 4$, $(-1)^n/n!$ is squeezed between $-1/2^n$ and $1/2^n$.

Figure 9.2

Limit of a Sequence

THEOREM 9.4 Absolute Value Theorem

For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0 \quad \text{then} \quad \lim_{n\to\infty} a_n = 0.$$

Pattern Recognition for Sequences

Pattern Recognition for Sequences

Sometimes the terms of a sequence are generated by some rule that does not explicitly identify the nth term of the sequence.

In such cases, you may be required to discover a *pattern* in the sequence and to describe the nth term.

Once the nth term has been specified, you can investigate the convergence or divergence of the sequence.

Example 6 – *Finding the nth Term of a Sequence*

Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \ldots$$

and then determine whether the particular sequence you have chosen converges or diverges.

Example 6 – *Solution*

First, note that the numerators are successive powers of 2, and the denominators form the sequence of positive odd integers.

By comparing a_n with n, you have the following pattern.

$$\frac{2^1}{1}, \frac{2^2}{3}, \frac{2^3}{5}, \frac{2^4}{7}, \frac{2^5}{9}, \ldots, \frac{2^n}{2n-1}, \ldots$$

Example 6 – *Solution* (cont.)

Consider the function of a real variable $f(x) = 2^x/(2x - 1)$. Applying L'Hôpital's Rule produces

$$\lim_{x\to\infty} \frac{2^x}{2x - 1} = \lim_{x\to\infty} \frac{2^x(\ln 2)}{2} = \infty.$$

Next, apply Theorem 9.1 to conclude that

$$\lim_{n\to\infty} \frac{2^n}{2n - 1} = \infty.$$

So, the sequence diverges.

Monotonic Sequences and Bounded Sequences

Monotonic Sequences and Bounded Sequences

Definition of Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** when its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdot \cdot \cdot \leq a_n \leq \cdot \cdot \cdot$$

or when its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdot \cdot \cdot \geq a_n \geq \cdot \cdot \cdot .$$

Example 8 – *Determining Whether a Sequence Is Monotonic*

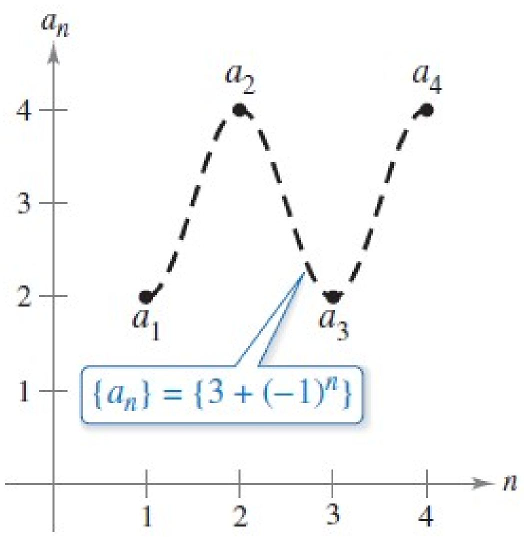
Determine whether each sequence having the given *n*th term is monotonic.

a. $a_n = 3 + (-1)^n$ **b.** $b_n = \dfrac{2n}{1+n}$ **c.** $c_n = \dfrac{n^2}{2^n - 1}$

Solution:

a. This sequence alternates between 2 and 4.

So, it is not monotonic.



(a) Not monotonic

Figure 9.3(a)

Example 8 – *Solution*

cont'd

b. This sequence is monotonic because each successive term is larger than its predecessor.

To see this, compare the terms b_n and b_{n+1}.

[Note that, because n is positive, you can multiply each side of the inequality by (1 + n) and (2 + n) without reversing the inequality sign.]

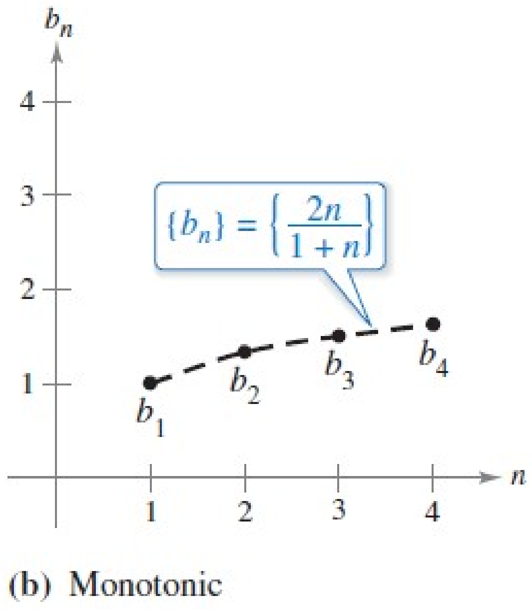


Figure 9.3(b)

Example 8 – *Solution*

cont'd

$$b_n = \frac{2n}{1+n} \overset{?}{<} \frac{2(n+1)}{1+(n+1)} = b_{n+1}$$

$$2n(2+n) \overset{?}{<} (1+n)(2n+2)$$

$$4n + 2n^2 \overset{?}{<} 2 + 4n + 2n^2$$

$$0 < 2$$

Starting with the final inequality, which is valid, you can reverse the steps to conclude that the original inequality is also valid.

Example 8 – *Solution*

cont'd

c. This sequence is not monotonic, because the second term is larger than the first term, and larger than the third.

(Note that if you drop the first term, the remaining sequence $c_2, c_3, c_4, \ldots$ is monotonic.)

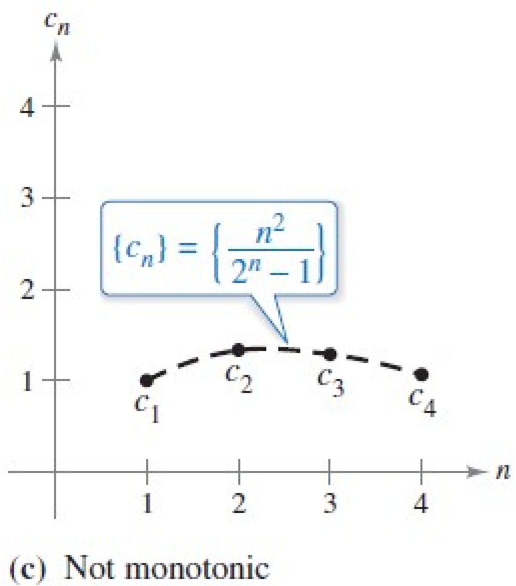


Figure 9.3(c)

Monotonic Sequences and Bounded Sequences

Definition of Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** when there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** when there is a real number N such that $N \leq a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** when it is bounded above and bounded below.

Monotonic Sequences and Bounded Sequences

One important property of the real numbers is that they are **complete.** Informally this means that there are no holes or gaps on the real number line. (The set of rational numbers does not have the completeness property.)

The completeness axiom for real numbers can be used to conclude that if a sequence has an upper bound, it must have a **least upper bound** (an upper bound that is smaller than all other upper bounds for the sequence).

Monotonic Sequences and Bounded Sequences

For example, the least upper bound of the sequence $\{a_n\} = \{n/(n + 1)\}$,

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$$

is 1.

THEOREM 9.5 Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Example 9 – *Bounded and Monotonic Sequences*

a. The sequence $\{a_n\} = \{1/n\}$ is both bounded and monotonic and so, by Theorem 9.5, must converge.

b. The divergent sequence $\{b_n\} = \{n^2/(n + 1)\}$ is monotonic, but not bounded. (It is bounded below.)

c. The divergent sequence $\{c_n\} = \{(–1)^n\}$ is bounded, but not monotonic.