8 Integration Techniques, L'Hôpital's Rule, and Improper Integrals

8.8 Improper Integrals

Objectives

- Evaluate an improper integral that has an infinite limit of integration.

- Evaluate an improper integral that has an infinite discontinuity.

Improper Integrals with Infinite Limits of Integration

Improper Integrals with Infinite Limits of Integration

The definition of a definite integral

$$\int_a^b f(x)\, dx$$

requires that the interval [*a*, *b*] be finite.

Furthermore, the Fundamental Theorem of Calculus requires that *f* be continuous on [*a*,*b*]/

You will study a procedure for evaluating integrals that do not satisfy these requirements—usually because either one or both of the limits of integration are infinite, or *f* has a finite number of infinite discontinuities in the interval [*a*, *b*].

Improper Integrals with Infinite Limits of Integration

Integrals that possess either property are **improper integrals.**

Note that a function *f* is said to have an **infinite discontinuity** at *c* if, *from the right or left,*

$$\lim_{x \to c} f(x) = \infty \qquad \text{or} \qquad \lim_{x \to c} f(x) = -\infty.$$

To get an idea of how to evaluate an improper integral, consider the integral

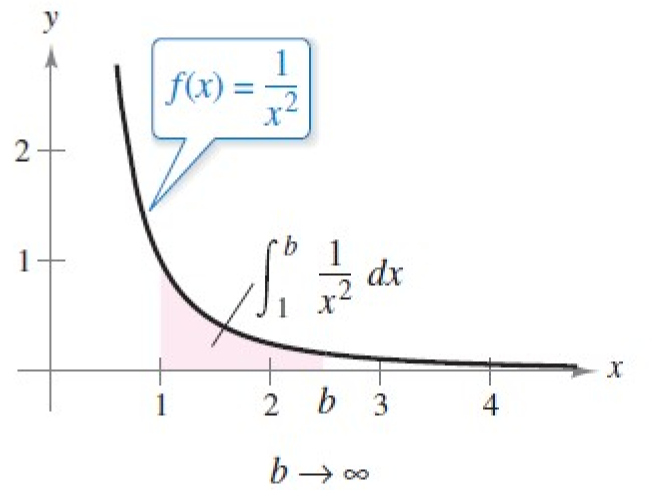
$$\int_1^b \frac{dx}{x^2} = -\frac{1}{x}\Big]_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$$

Improper Integrals with Infinite Limits of Integration

This integral can be interpreted as the area of the shaded region shown in Figure 8.17. Taking the limit as $b \rightarrow \infty$ produces

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty}\left(\int_1^b \frac{dx}{x^2}\right) = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1.$$

This improper integral can be interpreted as the area of the *unbounded* region between the graph of $f(x) = 1/x^2$ and the x-axis (to the right of $x = 1$.)



The unbounded region has an area of 1.

Figure 8.17

Improper Integrals with Infinite Limits of Integration

Definition of Improper Integrals with Infinite Integration Limits

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

where c is any real number (see Exercise 111).

In the first two cases, the improper integral **converges** when the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

Example 1 – *An Improper Integral That Diverges*

Evaluate $\displaystyle\int_1^\infty \frac{dx}{x}$.

Solution:

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} \qquad \text{Take limit as } b \to \infty.$$

$$= \lim_{b\to\infty} \Big[\ln x \Big]_1^b \qquad \text{Apply Log Rule.}$$

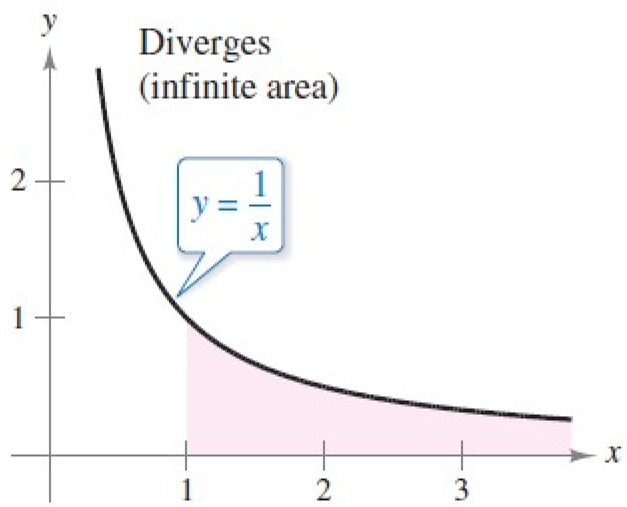
$$= \lim_{b\to\infty} (\ln b - 0) \qquad \text{Apply Fundamental Theorem of Calculus.}$$

$$= \infty \qquad \text{Evaluate limit.}$$

Example 1 – *Solution*

cont'd

See Figure 8.18.



This unbounded region has an infinite area.

Figure 8.18

Improper Integrals with Infinite Discontinuities

Improper Integrals with Infinite Discontinuities

The second basic type of improper integral is one that has an infinite discontinuity *at or between* the limits of integration.

Definition of Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In the first two cases, the improper integral **converges** when the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

Example 6 – *An Improper Integral with an Infinite Discontinuity*

Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}.$

Solution:

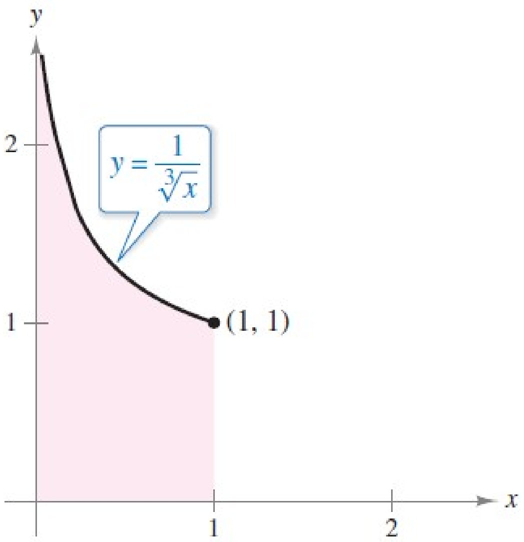
The integrand has an infinite discontinuity at $x = 0$, as shown in Figure 8.23.

You can evaluate this integral as shown below

$$\int_0^1 x^{-1/3}\,dx = \lim_{b\to 0^+}\left[\frac{x^{2/3}}{2/3}\right]_b^1$$

$$= \lim_{b\to 0^+}\frac{3}{2}\left(1 - b^{2/3}\right)$$

$$= \frac{3}{2}$$



Infinite discontinuity at $x = 0$

Figure 8.23

Improper Integrals with Infinite Discontinuities

THEOREM 8.5 A Special Type of Improper Integral

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \text{diverges}, & p \leq 1 \end{cases}$$

Example 11 – *An Application Involving A Solid of Revolution*

The solid formed by revolving (about the x-axis) the *unbounded* region lying between the graph of $f(x) = 1/x$ and the x-axis ($x \geq 1$) is called **Gabriel's Horn.** (See Figure 8.27.) Show that this solid has a finite volume and an infinite surface area.

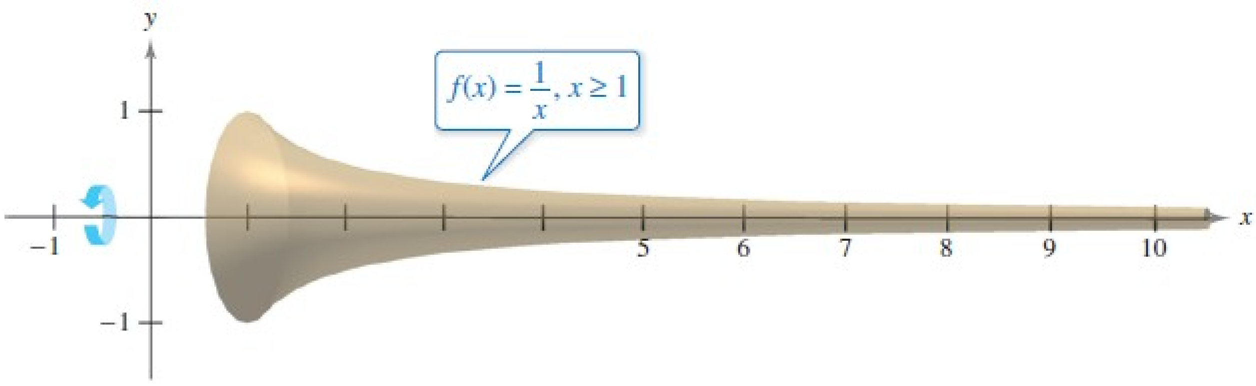


Figure 8.27

Example 11 – *Solution*

Using the disk method and Theorem 8.5, you can determine the volume to be

$$V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx \qquad \text{Theorem 8.5, } p = 2 > 1$$

$$= \pi\left(\frac{1}{2-1}\right) = \pi.$$

The surface area is given by

$$S = 2\pi \int_1^\infty f(x)\sqrt{1 + [f'(x)]^2}\, dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx.$$

Example 11 – *Solution*

cont'd

Because

$$\sqrt{1 + \frac{1}{x^4}} > 1$$

on the interval [1, ∞), and the improper integral

$$\int_1^{\infty} \frac{1}{x}\,dx$$

diverges, you can conclude that the improper integral

$$\int_1^{\infty} \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx$$

also diverges.

So, the surface area is infinite.