8 Integration Techniques, L'Hôpital's Rule, and Improper Integrals

8.7 Indeterminate Forms and L'Hôpital's Rule

Objectives

- Recognize limits that produce indeterminate forms.
- Apply L'Hôpital's Rule to evaluate a limit.

Indeterminate Forms

Indeterminate Forms

The forms 0/0 and ∞/∞ are called *indeterminate* because they do not guarantee that a limit exists, nor do they indicate what the limit is, if one does exist.

When you encountered one of these indeterminate forms earlier in the text, you attempted to rewrite the expression by using various algebraic techniques.

Indeterminate Form	Limit	Algebraic Technique
$\frac{0}{0}$	$\lim_{x\to-1}\frac{2x^2-2}{x+1}=\lim_{x\to-1}2(x-1)$ $=-4$	Divide numerator and denominator by $(x+1)$.
$\frac{\infty}{\infty}$	$\lim_{x\to\infty}\frac{3x^2-1}{2x^2+1}=\lim_{x\to\infty}\frac{3-(1/x^2)}{2+(1/x^2)}$ $=\frac{3}{2}$	Divide numerator and denominator by x^2.

Indeterminate Forms

Occasionally, you can extend these algebraic techniques to find limits of transcendental functions. For instance, the limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{e^x - 1}$$

produces the indeterminate form 0/0. Factoring and then dividing produces

$$\lim_{x \to 0} \frac{e^{2x} - 1}{e^x - 1} = \lim_{x \to 0} \frac{(e^x + 1)(e^x - 1)}{e^x - 1}$$

$$= \lim_{x \to 0} (e^x + 1)$$

$$= 2.$$

Indeterminate Forms

Not all indeterminate forms, however, can be evaluated by algebraic manipulation. This is often true when *both* algebraic and transcendental functions are involved.

For instance, the limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}$$

produces the indeterminate form 0/0.

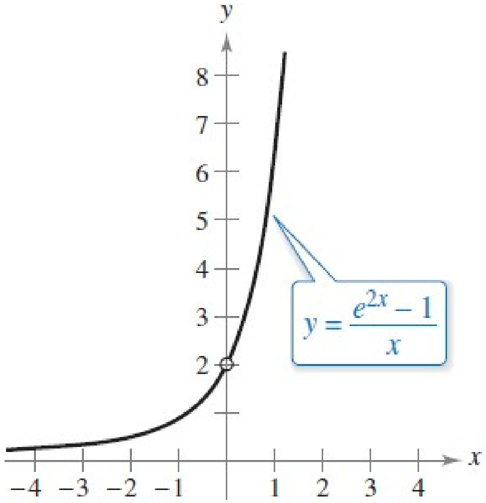
Rewriting the expression to obtain

$$\lim_{x \to 0} \left(\frac{e^{2x}}{x} - \frac{1}{x} \right)$$

merely produces another indeterminate form, $\infty - \infty$.

Indeterminate Forms

You could use technology to estimate the limit, as shown in the table and in Figure 8.15. From the table and the graph, the limit appears to be 2.



The limit as x approaches 0 appears to be 2.

Figure 8.15

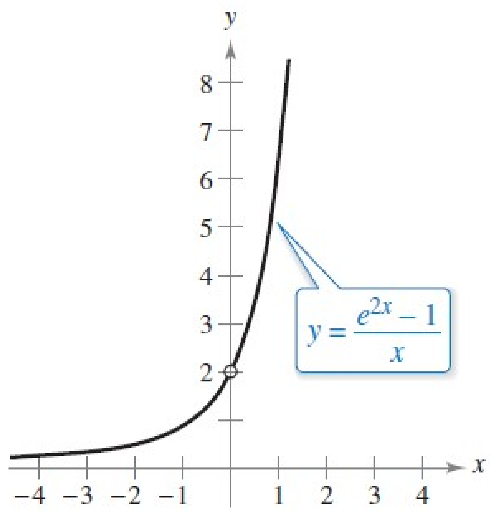
x	-1	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	1
$\frac{e^{2x} - 1}{x}$	0.865	1.813	1.980	1.998	?	2.002	2.020	2.214	6.389

L’Hôpital’s Rule

L’Hôpital’s Rule

To find the limit illustrated in Figure 8.15, you can use a theorem called **L’Hôpital’s Rule**. This theorem states that under certain conditions, the limit of the quotient $f(x)/g(x)$ is determined by the limit of the quotient of the derivatives

$$\frac{f'(x)}{g'(x)}.$$



The limit as x approaches 0 appears to be 2.

Figure 8.15

L'Hôpital's Rule

To prove this theorem, you can use a more general result called the **Extended Mean Value Theorem**.

THEOREM 8.3 The Extended Mean Value Theorem

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

L'Hôpital's Rule

THEOREM 8.4 L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

Example 1 – *Indeterminate Form 0/0*

Evaluate $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$.

Solution:

Because direct substitution results in the indeterminate form 0/0.

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}$$

$$\lim_{x \to 0} (e^{2x} - 1) = 0$$

$$\lim_{x \to 0} x = 0$$

Example 1 – *Solution*

cont'd

You can apply L'Hôpital's Rule, as shown below.

$$\lim_{x\to 0} \frac{e^{2x} - 1}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}[e^{2x} - 1]}{\frac{d}{dx}[x]}$$

Apply L'Hôpital's Rule.

$$= \lim_{x\to 0} \frac{2e^{2x}}{1}$$

Differentiate numerator and denominator.

$$= 2$$

Evaluate the limit.

L’Hôpital’s Rule

The forms $0/0, \infty/\infty, \infty - \infty, 0 \cdot \infty, 0^0, 1^\infty$, and ∞^0 have been identified as *indeterminate*. There are similar forms that you should recognize as “determinate.”

$\infty + \infty \to \infty$	Limit is positive infinity.
$-\infty - \infty \to -\infty$	Limit is negative infinity.
$0^\infty \to 0$	Limit is zero.
$0^{-\infty} \to \infty$	Limit is positive infinity.

As a final comment, remember that L’Hôpital’s Rule can be applied only to quotients leading to the indeterminate forms 0/0 and ∞/∞.

L’Hôpital’s Rule

For instance, the following application of L’Hôpital’s Rule is *incorrect*.

$$\lim_{x \to 0} \frac{e^x}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1$$

Incorrect use of L’Hôpital’s Rule

The reason this application is incorrect is that, even though the limit of the denominator is 0, the limit of the numerator is 1, which means that the hypotheses of L’Hôpital’s Rule have not been satisfied.