8 Integration Techniques, L'Hôpital's Rule, and Improper Integrals

8.5 Partial Fractions

Objectives

- Understand the concept of partial fraction decomposition.
- Use partial fraction decomposition with linear factors to integrate rational functions.
- Use partial fraction decomposition with quadratic factors to integrate rational functions.

Partial Fractions

Partial Fractions

The **method of partial fractions** is a procedure for decomposing a rational function into simpler rational functions to which you can apply the basic integration formulas.

To see the benefit of the method of partial fractions, consider the integral

$$\int \frac{1}{x^2 - 5x + 6}\, dx.$$

Partial Fractions

To evaluate this integral *without* partial fractions, you can complete the square and use trigonometric substitution (see Figure 8.13) to obtain

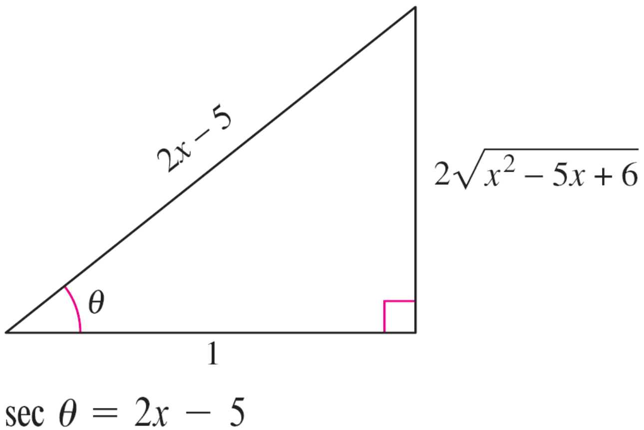

Figure 8.13

$$\int \frac{1}{x^2-5x+6}\,dx = \int \frac{dx}{(x-5/2)^2-(1/2)^2}$$

$$a = \tfrac{1}{2},\; x - \tfrac{5}{2} = \tfrac{1}{2}\sec\theta$$

$$= \int \frac{(1/2)\sec\theta\tan\theta\,d\theta}{(1/4)\tan^2\theta}$$

$$dx = \tfrac{1}{2}\sec\theta\tan\theta\,d\theta$$

$$= 2\int \csc\theta\,d\theta$$

Partial Fractions

$$
\begin{aligned}
&= 2\ln\left|\csc\theta - \cot\theta\right| + C \\
&= 2\ln\left|\frac{2x-5}{2\sqrt{x^2-5x+6}} - \frac{1}{2\sqrt{x^2-5x+6}}\right| + C \\
&= 2\ln\left|\frac{x-3}{\sqrt{x^2-5x+6}}\right| + C \\
&= 2\ln\left|\frac{\sqrt{x-3}}{\sqrt{x-2}}\right| + C \\
&= \ln\left|\frac{x-3}{x-2}\right| + C \\
&= \ln|x-3| - \ln|x-2| + C.
\end{aligned}
$$

Partial Fractions

Now, suppose you had observed that

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x-3} - \frac{1}{x-2}.$$

Partial fraction decomposition

Then you could evaluate the integral easily, as follows.

$$\int \frac{1}{x^2 - 5x + 6}\,dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx$$
$$= \ln|x-3| - \ln|x-2| + C$$

This method is clearly preferable to trigonometric substitution. However, its use depends on the ability to factor the denominator. $x^2 - 5x + 6$. and to find the **partial fractions** $\frac{1}{x-3}$ and $-\frac{1}{x-2}$.

Partial Fractions

Recall from algebra that every polynomial with real coefficients can be factored into linear and irreducible quadratic factors.

For instance, the polynomial

$$x^5 + x^4 - x - 1$$

can be written as

$$\begin{aligned} x^5 + x^4 - x - 1 &= x^4(x + 1) - (x + 1) \\ &= (x^4 - 1)(x + 1) \\ &= (x^2 + 1)(x^2 - 1)(x + 1) \\ &= (x^2 + 1)(x + 1)(x - 1)(x + 1) \\ &= (x - 1)(x + 1)^2(x^2 + 1) \end{aligned}$$

Partial Fractions

where $(x – 1)$ is a linear factor, $(x + 2)^2$ is a repeated linear factor, and $(x^2 + 1)$ is an irreducible quadratic factor.

Using this factorization, you can write the partial fraction decomposition of the rational expression

$$\frac{N(x)}{x^5 + x^4 - x - 1}$$

where $N(x)$ is a polynomial of degree less than 5, as shown.

$$\frac{N(x)}{(x - 1)(x + 1)^2(x^2 + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} + \frac{Dx + E}{x^2 + 1}$$

Partial Fractions

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. **Divide when improper:** When $N(x)/D(x)$ is an improper fraction (that is, when the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px + q)^m \quad \text{and} \quad (ax^2 + bx + c)^n$$

where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Linear Factors

Example 1 – *Distinct Linear Factors*

Write the partial fraction decomposition for $\frac{1}{x^2 - 5x + 6}$.

Solution:

Because $x^2 - 5x + 6 = (x - 3)(x - 2)$, you should include one partial fraction for each factor and write

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x - 3} + \frac{B}{x - 2}$$

where *A* and *B* are to be determined.

Multiplying this equation by the least common denominator $(x - 3)(x - 2)$ yields the **basic equation**

$1 = A(x - 2) + B(x - 3)$. Basic equation.

Example 1 – *Solution*

cont'd

Because this equation is to be true for all x, you can substitute any *convenient* values for x to obtain equations in A and B.

The most convenient values are the ones that make particular factors equal to 0.

To solve for A, let $x = 3$.

$$1 = A(3 - 2) + B(3 - 3)$$

Let $x = 3$ in basic equation.

$$1 = A(1) + B(0)$$

$$1 = A$$

Example 1 – *Solution*

cont'd

To solve for B, let $x = 2$ and obtain

$$1 = A(2 - 2) + B(2 - 3)$$

Let $x = 2$ in basic equation

$$1 = A(0) + B(-1)$$

$$-1 = B$$

So, the decomposition is

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{x - 3} - \frac{1}{x - 2}$$

as shown at the beginning of this section.

Quadratic Factors

Example 3 – *Distinct Linear and Quadratic Factors*

Find $\displaystyle\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)}\,dx.$

Solution:

Because $(x^2 - x)(x^2 + 4) = x(x - 1)(x^2 + 4)$ you should include one partial fraction for each factor and write

$$\frac{2x^3 - 4x - 8}{x(x - 1)(x^2 + 4)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 4}.$$

Multiplying by the least common denominator $x(x - 1)(x^2 + 4)$ yields the *basic equation*

$2x^3 - 4x - 8 = A(x - 1)(x^2 + 4) + Bx(x^2 + 4) + (Cx + D)(x)(x - 1).$

Example 3 – *Solution*

cont'd

To solve for A, let $x = 0$ and obtain

$$-8 = A(-1)(4) + 0 + 0$$

$$2 = A$$

To solve for B, let $x = 1$ and obtain

$$-10 = 0 + B(5) + 0$$

$$-2 = B$$

At this point, C and D are yet to be determined. You can find these remaining constants by choosing two other values for x and solving the resulting system of linear equations.

Example 3 – *Solution*

cont'd

Using $x = -1$, then, $A = 2$ and $B = -2$, you can write

$$-6 = (2)(-2)(5) + (-2)(-1)(5) + (-C + D)(-1)(-2)$$

$$2 = -C + D$$

If $x = 2$, you have

$$0 = (2)(1)(8) + (-2)(2)(8) + (2C + D)(2)(1)$$

$$8 = 2C + D$$

Solving the linear system by subtracting the first equation from the second

$$-C + D = 2$$

$$2C + D = 8$$

yields $C = 2$

Example 3 – *Solution*

cont'd

Consequently, $D = 4$, and it follows that

$$\int \frac{2x^3 - 4x - 8}{x(x-1)(x^2+4)}\,dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\right)dx$$

$$= 2\ln|x| - 2\ln|x-1| + \ln(x^2+4) + 2\arctan\frac{x}{2} + C.$$

Quadratic Factors

Here are some guidelines for solving the basic equation that is obtained in a partial fraction decomposition.

GUIDELINES FOR SOLVING THE BASIC EQUATION

Linear Factors

1. Substitute the roots of the distinct linear factors in the basic equation.
2. For repeated linear factors, use the coefficients determined in the first guideline to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x.
3. Equate the coefficients of like powers to obtain a system of linear equations involving A, B, C, and so on.
4. Solve the system of linear equations.