8 Integration Techniques, L’Hôpital’s Rule, and Improper Integrals

8.4 Trigonometric Substitution

Objectives

- Use trigonometric substitution to solve an integral.

- Use integrals to model and solve real-life applications.

Trigonometric Substitution

Trigonometric Substitution

You can ise **trigonometric substitution** to evaluate integrals involving the radicals

$$\sqrt{a^2 - u^2}, \qquad \sqrt{a^2 + u^2}, \qquad \text{and} \qquad \sqrt{u^2 - a^2}.$$

The objective with trigonometric substitution is to eliminate the radical in the integrand. You do this by using the Pythagorean identities

$$\cos^2\theta = 1 - \sin^2\theta$$
$$\sec^2\theta = 1 + \tan^2\theta$$
$$\tan^2\theta = \sec^2\theta - 1$$

Trigonometric Substitution

For example, for $a > 0$, let $u = a\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then

$$\begin{aligned}\sqrt{a^2 - u^2} &= \sqrt{a^2 - a^2\sin^2\theta}\\ &= \sqrt{a^2(1 - \sin^2\theta)}\\ &= \sqrt{a^2\cos^2\theta}\\ &= a\cos\theta.\end{aligned}$$

Note that $\cos\theta \geq 0$, because $-\pi/2 \leq \theta \leq \pi/2$.

Trigonometric Substitution

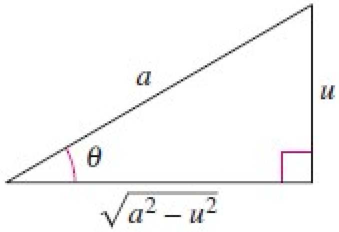
Trigonometric Substitution ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

   $u = a \sin \theta.$

   Then $\sqrt{a^2 - u^2} = a \cos \theta$, where

   $-\pi/2 \le \theta \le \pi/2.$

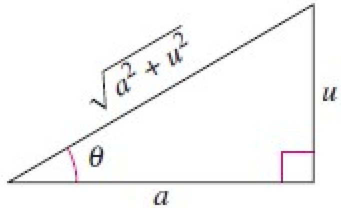


2. For integrals involving $\sqrt{a^2 + u^2}$, let

   $u = a \tan \theta.$

   Then $\sqrt{a^2 + u^2} = a \sec \theta$, where

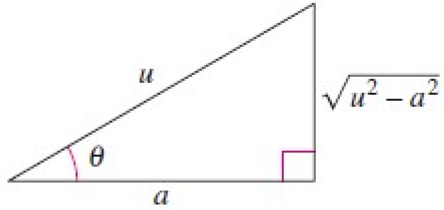
   $-\pi/2 < \theta < \pi/2.$



3. For integrals involving $\sqrt{u^2 - a^2}$, let

   $u = a \sec \theta.$

   Then



$$\sqrt{u^2 - a^2} = \begin{cases} a \tan \theta \text{ for } u > a, \text{ where } 0 \le \theta \le \pi/2 \\ -a \tan \theta \text{ for } u < -a, \text{ where } \pi/2 < \theta \le \pi. \end{cases}$$

Example 1 – *Trigonometric Substitution: u = asin* θ

Find $\displaystyle\int \frac{dx}{x^2\sqrt{9 - x^2}}.$

Solution:

First, note that none of the basic integration rules applies.

To use trigonometric substitution, you should observe that $\sqrt{9 - x^2}$ is of the form $\sqrt{a^2 - u^2}$.

So, you can use the substitution

$$x = a \sin \theta = 3 \sin \theta.$$

Example 1 – *Solution*

cont'd

Using differentiation and the triangle shown in Figure 8.6, you obtain

$$dx = 3\cos\theta\, d\theta, \qquad \sqrt{9 - x^2} = 3\cos\theta, \qquad \text{and} \qquad x^2 = 9\sin^2\theta.$$

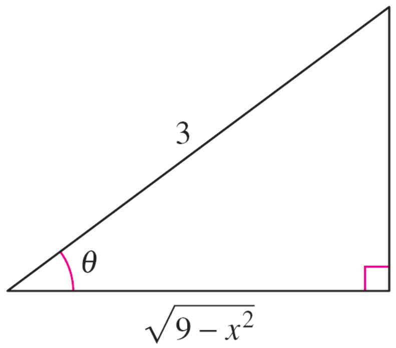
So, trigonometric substitution yields

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta\, d\theta}{(9\sin^2\theta)(3\cos\theta)} \qquad \text{Substitute.}$$

$$= \frac{1}{9}\int \frac{d\theta}{\sin^2\theta} \qquad \text{Simplify.}$$

$$= \frac{1}{9}\int \csc^2\theta\, d\theta \qquad \text{Trigonometric identity}$$

$$= -\frac{1}{9}\cot\theta + C \qquad \text{Apply Cosecant Rule.}$$



$$\sin\theta = \frac{x}{3},\ \cot\theta = \frac{\sqrt{9-x^2}}{x}$$

Figure 8.6

Example 1 – *Solution*

cont'd

$$= -\frac{1}{9}\left(\frac{\sqrt{9 - x^2}}{x}\right) + C \qquad \text{Substitute for } \cot\theta.$$

$$= -\frac{\sqrt{9 - x^2}}{9x} + C.$$

Note that the triangle in Figure 8.6 can be used to convert the θ's back to x's, as follows.

$$\cot\theta = \frac{\text{adj.}}{\text{opp.}}$$

$$= \frac{\sqrt{9 - x^2}}{x}$$

Trigonometric Substitution

Trigonometric substitution can be used to evaluate the three integrals listed in the next theorem. These integrals will be encountered several times.

THEOREM 8.2 Special Integration Formulas ($a > 0$)

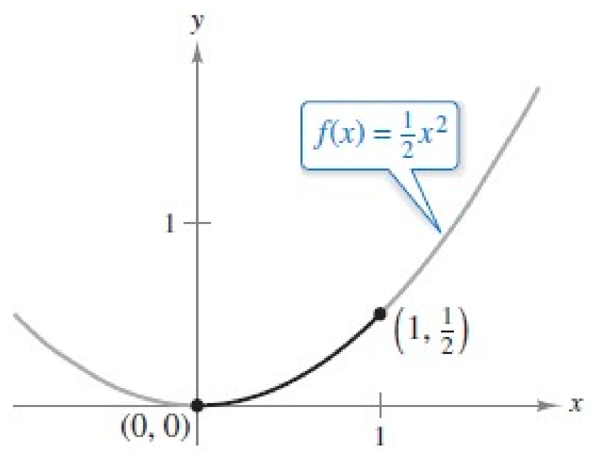
1. $\int \sqrt{a^2 - u^2}\, du = \frac{1}{2}\left(a^2 \arcsin \frac{u}{a} + u\sqrt{a^2 - u^2}\right) + C$
2. $\int \sqrt{u^2 - a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln\left|u + \sqrt{u^2 - a^2}\right|\right) + C, \quad u > a$
3. $\int \sqrt{u^2 + a^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 + a^2} + a^2 \ln\left|u + \sqrt{u^2 + a^2}\right|\right) + C$

Applications

Example 5 – *Finding Arc Length*

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from

x = 0 to *x* = 1 (see Figure 8.10).



The arc length of the curve from (0, 0) to $\left(1, \frac{1}{2}\right)$

Figure 8.10

Example 5 – *Solution*

Refer to the arc length formula.

$$s = \int_0^1 \sqrt{1 + [f'(x)]^2}\, dx \qquad \text{Formula for arc length}$$

$$= \int_0^1 \sqrt{1 + x^2}\, dx \qquad f'(x) = x$$

$$= \int_0^{\pi/4} \sec^3 \theta \, d\theta \qquad \text{Let } a = 1 \text{ and } x = \tan \theta.$$

$$= \frac{1}{2}\Big[\sec \theta \tan \theta + \ln|\sec \theta + \tan \theta|\Big]_0^{\pi/4} \qquad \text{Example 5, Section 8.2}$$

$$= \frac{1}{2}\left[\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right]$$

$$\approx 1.148$$