8 Integration Techniques, L’Hôpital’s Rule, and Improper Integrals

8.3 Trigonometric Integrals

Objectives

- Solve trigonometric integrals involving powers of sine and cosine.
- Solve trigonometric integrals involving powers of secant and tangent.
- Solve trigonometric integrals involving sine-cosine products with different angles.

Integrals Involving Powers of Sine and Cosine

Integrals Involving Powers of Sine and Cosine

In this section you will study techniques for evaluating integrals of the form

$$\int \sin^m x \cos^n x \, dx \qquad \text{and} \qquad \int \sec^m x \tan^n x \, dx$$

where either *m* or *n* is a positive integer.

To find antiderivatives for these forms, try to break them into combinations of trigonometric integrals to which you can apply the Power Rule.

Integrals Involving Powers of Sine and Cosine

For instance, you can evaluate $\int \sin^5 x \cos x \, dx$ with the Power Rule by letting $u = \sin x$. Then, $du = \cos x \, dx$ and you have

$$\int \sin^5 x \cos x \, dx = \int u^5 \, du = \frac{u^6}{6} + C = \frac{\sin^6 x}{6} + C.$$

To break up $\int \sin^m x \cos^n x \, dx$ into forms to which you can apply the Power Rule, use the following identities.

$\sin^2 x + \cos^2 x = 1$	Pythagorean identity
$\sin^2 x = \dfrac{1 - \cos 2x}{2}$	Half-angle identity for $\sin^2 x$
$\cos^2 x = \dfrac{1 + \cos 2x}{2}$	Half-angle identity for $\cos^2 x$

Integrals Involving Powers of Sine and Cosine

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SINE AND COSINE

1. When the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then, expand and integrate.

$$\int \sin^{\overbrace{2k+1}^{\text{Odd}}} x \cos^n x\, dx = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \cos^n x \overbrace{\sin x\, dx}^{\text{Save for } du} = \int (1 - \cos^2 x)^k \cos^n x \sin x\, dx$$

2. When the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then, expand and integrate.

$$\int \sin^m x \cos^{\overbrace{2k+1}^{\text{Odd}}} x\, dx = \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to sines}} \overbrace{\cos x\, dx}^{\text{Save for } du} = \int \sin^m x\, (1 - \sin^2 x)^k \cos x\, dx \frac{1}{2}$$

3. When the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in the second guideline.

Example 1 – *Power of Sine Is Odd and Positive*

Find $\int \sin^3 x \cos^4 x \, dx.$

Solution:

Because you expect to use the Power Rule with *u* = cos *x*, *save one sine factor* to form *du* and convert the remaining sine factors to cosines.

$$\begin{aligned}
\int \sin^3 x \cos^4 x \, dx &= \int \sin^2 x \cos^4 x (\sin x) \, dx && \text{Rewrite.} \\
&= \int (1 - \cos^2 x) \cos^4 x \sin x \, dx && \text{Trigonometric identity} \\
&= \int (\cos^4 x - \cos^6 x) \sin x \, dx && \text{Multiply.}
\end{aligned}$$

Example 1 – *Solution*

cont'd

$$= \int \cos^4 x \sin x \, dx - \int \cos^6 x \sin x \, dx$$ Rewrite.

$$= -\int \cos^4 x(-\sin x) \, dx + \int \cos^6 x(-\sin x) \, dx$$

$$= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$$ Integrate.

Integrals Involving Powers of Sine and Cosine

Wallis's Formulas

1. If n is odd $(n \geq 3)$, then

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) \cdots \left(\frac{n-1}{n}\right).$$

2. If n is even $(n \geq 2)$, then

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right).$$

These formulas are also valid if $\cos^n x$ is replaced by $\sin^n x$.

Integrals Involving Powers of Secant and Tangent

Integrals Involving Powers of Secant and Tangent

The following guidelines can help you evaluate integrals of the form $\int \sec^m x \tan^n x\, dx.$

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SECANT AND TANGENT

1. When the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then, expand and integrate.

$$\int \sec^{\overbrace{2k}^{\text{Even}}} x \tan^n x\, dx = \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x\, dx}^{\text{Save for } du} = \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x\, dx$$

2. When the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then, expand and integrate.

$$\int \sec^m x \tan^{\overbrace{2k+1}^{\text{Odd}}} x\, dx = \int \sec^{m-1} x \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \overbrace{\sec x \tan x\, dx}^{\text{Save for } du} = \int \sec^{m-1} x (\sec^2 x - 1)^k \sec x \tan x\, dx$$

Integrals Involving Powers of Secant and Tangent

cont'd

3. When there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x\,dx = \int \tan^{n-2} x\overbrace{(\tan^2 x)}^{\text{Convert to secants}}\,dx = \int \tan^{n-2} x(\sec^2 x - 1)\,dx$$

4. When the integral is of the form

$$\int \sec^m x\,dx$$

where m is odd and positive, use integration by parts, as illustrated in Example 5 in Section 8.2.

5. When none of the first four guidelines applies, try converting to sines and cosines.

Example 4 – *Power of Tangent Is Odd and Positive*

Find $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx.$

Solution:

Because you expect to use the Power Rule with *u* = sec *x*, *save a factor of* (*sec x tan x*) to form *du* and convert the remaining tangent factors to secants.

$$\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx = \int (\sec x)^{-1/2} \tan^3 x\,dx$$

$$= \int (\sec x)^{-3/2}(\tan^2 x)(\sec x \tan x)\,dx$$

$$= \int (\sec x)^{-3/2}(\sec^2 x - 1)(\sec x \tan x)\,dx$$

Example 4 – *Solution*

cont'd

$$= \int [(\sec x)^{1/2} - (\sec x)^{-3/2}](\sec x \tan x)\, dx$$

$$= \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C$$

Integrals Involving Sine-Cosine Products with Different Angles

Integrals Involving Sine-Cosine Products with Different Angles

Integrals involving the products of sines and cosines of two *different* angles occur in many applications.

In such instances, you can use the following product-to-sum identities.

$$\sin mx \sin nx = \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x])$$

$$\sin mx \cos nx = \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x])$$

$$\cos mx \cos nx = \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])$$

Example 8 – *Using Product-to-Sum Identities*

Find $\int \sin 5x \cos 4x \, dx.$

Solution:

Considering the second product-to-sum identity above, you can write

$$\int \sin 5x \cos 4x \, dx = \frac{1}{2}\int (\sin x + \sin 9x) \, dx$$

$$= \frac{1}{2}\left(-\cos x - \frac{\cos 9x}{9}\right) + C$$

$$= -\frac{\cos x}{2} - \frac{\cos 9x}{18} + C.$$