# 8 Integration Techniques, L'Hôpital's Rule, and Improper Integrals

# 8.2 Integration by Parts

# Objectives

- Find an antiderivative using integration by parts.

# Integration by Parts

# Integration by Parts

In this section you will study an important integration technique called **integration by parts.** This technique can be applied to a wide variety of functions and is particularly useful for integrands involving *products* of algebraic and transcendental functions.

For instance, integration by parts works well with integrals such as

$$\int x \ln x \, dx, \quad \int x^2 e^x \, dx, \quad \text{and} \quad \int e^x \sin x \, dx.$$

# Integration by Parts

Integration by parts is based on the formula for the derivative of a product

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= uv' + vu'$$

where both *u* and *v* are differentiable functions of *x.*

# Integration by Parts

When *u′* and *v′* are continuous, you can integrate both sides of the equation to obtain

$$uv = \int uv'\,dx + \int vu'\,dx$$

$$= \int u\,dv + \int v\,du.$$

By rewriting this equation, you obtain the next theorem.

# Integration by Parts

**THEOREM 8.1 Integration by Parts**

If $u$ and $v$ are functions of $x$ and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du.$$

The formula expresses the original integral in terms of another integral. Depending upon the choices of *u* and *dv*, it may be easier to evaluate the second integral than the original one.

# Integration by Parts

Because the choices of *u* and *dv* are critical in the integration by parts process, the guidelines below are provided.

**GUIDELINES FOR INTEGRATION BY PARTS**

1. Try letting $dv$ be the most complicated portion of the integrand that fits a basic integration rule. Then $u$ will be the remaining factor(s) of the integrand.
2. Try letting $u$ be the portion of the integrand whose derivative is a function simpler than $u$. Then $dv$ will be the remaining factor(s) of the integrand.

Note that $dv$ always includes the $dx$ of the original integrand.

# Integration by Parts

When using integration by parts, note that you can first choose $dv$ or first choose $u$.

After you choose, however, the choice of the other factor is determined– it must be the remaining portion of the integrand.

Also note that $dv$ must contain the differential $dx$ of the original integral.

# Example 1 – *Integration by Parts*

Find $\int xe^x\,dx.$

Solution:

To apply integration by parts, you need to write the integral in the form $\int u\,dv.$

There are several ways to do this.

$$\int \underbrace{(x)}_{u}\,\underbrace{(e^x\,dx)}_{dv}, \quad \int \underbrace{(e^x)}_{u}\underbrace{(x\,dx)}_{dv}, \quad \int \underbrace{(1)}_{u}\,\underbrace{(xe^x\,dx)}_{dv}, \quad \int \underbrace{(xe^x)}_{u}\underbrace{(dx)}_{dv}$$

The guidelines suggest the first option because the derivative of $u = x$ is simpler than $x$, and $dv = e^x\,dx$ is the most complicated portion of the integrand that fits a basic integration formula.

# Example 1 – *Solution*

cont'd

$$dv = e^x\,dx \quad \Rightarrow \quad v = \int dv = \int e^x\,dx = e^x$$

$$u = x \quad \Rightarrow \quad du = dx$$

Now, integration by parts produces

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx \qquad \text{Substitute.}$$

$$= xe^x - e^x + C. \qquad \text{Integrate.}$$

To check this, differentiate $xe^x - e^x + C$ to see that you obtain the original integrand.

# Integration by Parts

As you gain experience in using integration by parts, your skill in determining $u$ and $dv$ will increase.

The next summary lists several common integrals with suggestions for the choices of $u$ and $v$.

# Integration by Parts

**SUMMARY: COMMON INTEGRALS USING INTEGRATION BY PARTS**

**1.** For integrals of the form

$$\int x^n e^{ax}\,dx, \quad \int x^n \sin ax\,dx, \quad \text{or} \quad \int x^n \cos ax\,dx$$

let $u = x^n$ and let $dv = e^{ax}\,dx$, $\sin ax\,dx$, or $\cos ax\,dx$.

**2.** For integrals of the form

$$\int x^n \ln x\,dx, \quad \int x^n \arcsin ax\,dx, \quad \text{or} \quad \int x^n \arctan ax\,dx$$

let $u = \ln x$, $\arcsin ax$, or $\arctan ax$ and let $dv = x^n\,dx$.

**3.** For integrals of the form

$$\int e^{ax} \sin bx\,dx \quad \text{or} \quad \int e^{ax} \cos bx\,dx$$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax}\,dx$.

# Integration by Parts

In problems involving repeated applications of integration by parts, a tabular method, illustrated in Example 7, can help to organize the work. This method works well for integrals of the form

$$\int x^n \sin ax\, dx, \quad \int x^n \cos ax\, dx, \quad \text{and} \quad \int x^n e^{ax}\, dx.$$

# Example 7 – *Using the Tabular Method*

Find $\int x^2 \sin 4x \, dx$.

Solution:

Begin as usual by letting $u = x^2$ and $dv = v' \, dx = \sin 4x \, dx$.
Next, create a table consisting of three columns, as shown.

| *Alternate Signs* | *u and Its Derivatives* | *v′ and Its Antiderivatives* |
|---|---|---|
| $+$ | $x^2$ | $\sin 4x$ |
| $-$ | $2x$ | $-\frac{1}{4}\cos 4x$ |
| $+$ | $2$ | $-\frac{1}{16}\sin 4x$ |
| $-$ | $0$ | $\frac{1}{64}\cos 4x$ |

Differentiate until you obtain 0 as a derivative.

# Example 7 – *Solution*

cont'd

The solution is obtained by adding the signed products of the diagonal entries:

$$\int x^2 \sin 4x\, dx = -\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32}\cos 4x + C.$$