

Integration Techniques, L'Hôpital's Rule, and Improper Integrals



Objectives

Find an antiderivative using integration by parts.

In this section you will study an important integration technique called **integration by parts.** This technique can be applied to a wide variety of functions and is particularly useful for integrands involving *products* of algebraic and transcendental functions.

For instance, integration by parts works well with integrals such as

$$\int x \ln x \, dx$$
, $\int x^2 \, e^x \, dx$, and $\int e^x \sin x \, dx$.

Integration by parts is based on the formula for the derivative of a product

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= uv' + vu'$$

where both u and v are differentiable functions of x.

When u' and v' are continuous, you can integrate both sides of the equation to obtain

$$uv = \int uv' dx + \int vu' dx$$
$$= \int u dv + \int v du.$$

By rewriting this equation, you obtain the next theorem.

THEOREM 8.1 Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int u \, dv = uv - \int v \, du.$$

The formula expresses the original integral in terms of another integral. Depending upon the choices of u and dv, it may be easier to evaluate the second integral than the original one.

Because the choices of *u* and *dv* are critical in the integration by parts process, the guidelines below are provided.

GUIDELINES FOR INTEGRATION BY PARTS

- 1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.
- 2. Try letting u be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand.

Note that dv always includes the dx of the original integrand.

When using integration by parts, note that you can first choose dv or first choose u.

After you choose, however, the choice of the other factor is determined—it must be the remaining portion of the integrand.

Also note that dv must contain the differential dx of the original integral.

Example 1 – Integration by Parts

Find
$$\int xe^x dx$$
.

Solution:

To apply integration by parts, you need to write the integral in the form $\int u \, dv$.

There are several ways to do this.

$$\int \underbrace{(x)}_{u} \underbrace{(e^{x} dx)}_{dv}, \quad \int \underbrace{(e^{x})}_{u} \underbrace{(x dx)}_{dv}, \quad \int \underbrace{(1)}_{u} \underbrace{(xe^{x} dx)}_{dv}, \quad \int \underbrace{(xe^{x})}_{u} \underbrace{(dx)}_{dv}$$

The guidelines suggest the first option because the derivative of u = x is simpler than x, and $dv = e^x dx$ is the most complicated portion of the integrand that fits a basic integration formula.

Example 1 – Solution

$$dv = e^x dx$$
 \Longrightarrow $v = \int dv = \int e^x dx = e^x$
 $u = x$ \Longrightarrow $du = dx$

Now, integration by parts produces

$$\int u \, dv = uv - \int v \, du$$
 Integration by parts formula
$$\int xe^x \, dx = xe^x - \int e^x \, dx$$
 Substitute.
$$= xe^x - e^x + C.$$
 Integrate.

To check this, differentiate $xe^x - e^x + C$ to see that you obtain the original integrand.

As you gain experience in using integration by parts, your skill in determining *u* and *dv* will increase.

The next summary lists several common integrals with suggestions for the choices of *u* and *v*.

SUMMARY: COMMON INTEGRALS USING INTEGRATION BY PARTS

1. For integrals of the form

$$\int x^n e^{ax} dx, \quad \int x^n \sin ax dx, \quad \text{or} \quad \int x^n \cos ax dx$$

let $u = x^n$ and let $dv = e^{ax} dx$, $\sin ax dx$, or $\cos ax dx$.

2. For integrals of the form

$$\int x^n \ln x \, dx, \quad \int x^n \arcsin ax \, dx, \quad \text{or} \quad \int x^n \arctan ax \, dx$$

let $u = \ln x$, arcsin ax, or arctan ax and let $dv = x^n dx$.

3. For integrals of the form

$$\int e^{ax} \sin bx \, dx \quad \text{or} \quad \int e^{ax} \cos bx \, dx$$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax} dx$.

In problems involving repeated applications of integration by parts, a tabular method, illustrated in Example 7, can help to organize the work. This method works well for integrals of the form

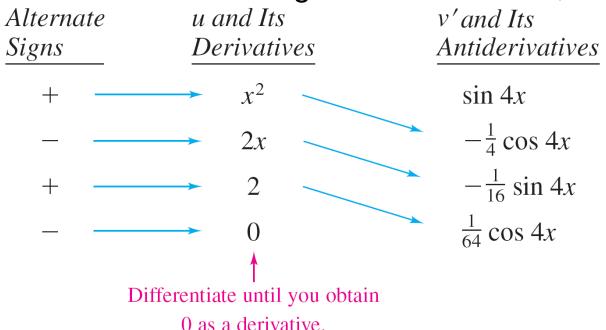
$$\int x^n \sin ax \, dx, \quad \int x^n \cos ax \, dx, \quad \text{and} \quad \int x^n \, e^{ax} \, dx.$$

Example 7 – Using the Tabular Method

Find
$$\int x^2 \sin 4x \, dx$$
.

Solution:

Begin as usual by letting $u = x^2$ and $dv = v' dx = \sin 4x dx$. Next, create a table consisting of three columns, as shown.



Example 7 – Solution

The solution is obtained by adding the signed products of the diagonal entries:

$$\int x^2 \sin 4x \, dx = -\frac{1}{4} x^2 \cos 4x + \frac{1}{8} x \sin 4x + \frac{1}{32} \cos 4x + C.$$