7 Applications of Integration

7.2 Volume: The Disk Method

Objectives

- Find the volume of a solid of revolution using the disk method.
- Find the volume of a solid of revolution using the washer method.
- Find the volume of a solid with known cross sections.

The Disk Method

The Disk Method

If a region in the plane is revolved about a line, the resulting solid is a **solid of revolution,** and the line is called the **axis of revolution.**

The simplest such solid is a right circular cylinder or **disk,** which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle,
as shown in Figure 7.13.

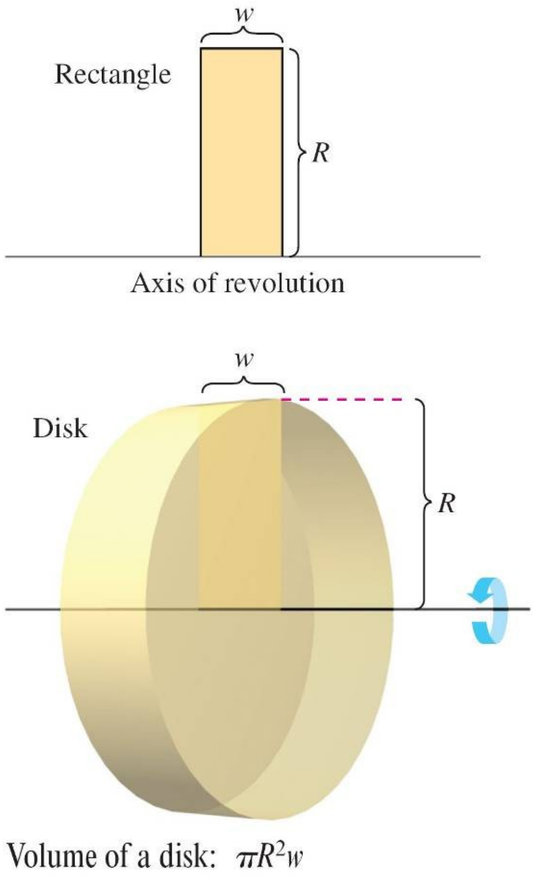


Figure 7.13

The Disk Method

The volume of such a disk is

$$\text{Volume of disk} = (\text{area of disk})(\text{width of disk})$$
$$= \pi R^2 w$$

where R is the radius of the disk and w is the width.

The Disk Method

To see how to use the volume of a disk to find the volume of a general solid of revolution, consider a solid of revolution formed by revolving the plane region in Figure 7.14 about the indicated axis.

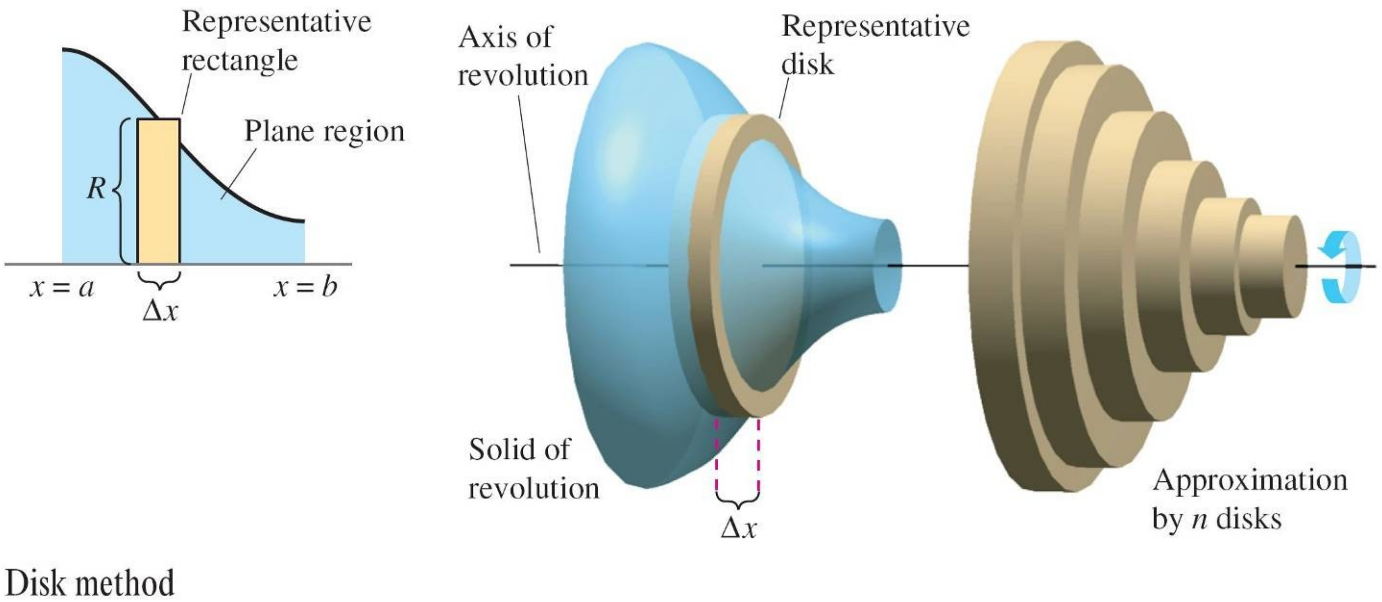


Figure 7.14

The Disk Method

To determine the volume of this solid, consider a representative rectangle in the plane region. When this rectangle is revolved about the axis of revolution, it generates a representative disk whose volume is

$$\Delta V = \pi R^2 \Delta x.$$

Approximating the volume of the solid by *n* such disks of width Δx and radius $R(x_i)$ produces

$$\text{Volume of solid} \approx \sum_{i=1}^{n} \pi [R(x_i)]^2 \Delta x$$

$$= \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x.$$

The Disk Method

This approximation appears to become better and better as $\|\Delta\| \to 0$ $(n \to \infty)$. So, you can define the volume of the solid as

$$\text{Volume of solid} = \lim_{\|\Delta\| \to 0} \pi \sum_{i=1}^{n} [R(x_i)]^2 \, \Delta x = \pi \int_a^b [R(x)]^2 \, dx.$$

Schematically, the disk method looks like this.

Known Precalculus Formula		**Representative Element**		**New Integration Formula**
Volume of disk $V = \pi R^2 w$	⇒	$\Delta V = \pi [R(x_i)]^2 \, \Delta x$	⇒	Solid of revolution $V = \pi \int_a^b [R(x)]^2 \, dx$

The Disk Method

A similar formula can be derived if the axis of revolution is vertical.

THE DISK METHOD

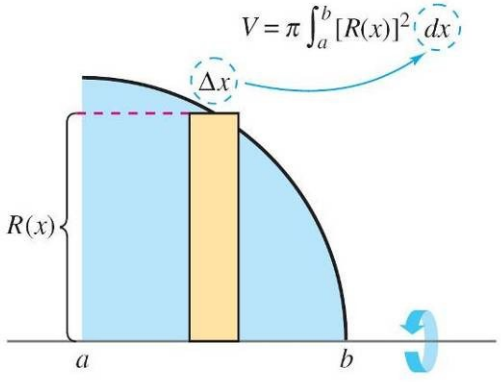
To find the volume of a solid of revolution with the **disk method,** use one of the formulas below. (See Figure 7.15.)

Horizontal Axis of Revolution

$$\text{Volume} = V = \pi \int_a^b [R(x)]^2\, dx$$

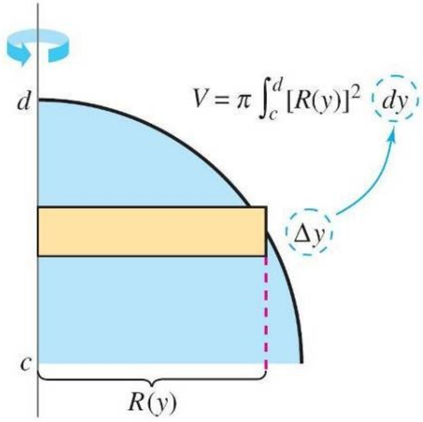
Vertical Axis of Revolution

$$\text{Volume} = V = \pi \int_c^d [R(y)]^2\, dy$$



Horizontal axis of revolution

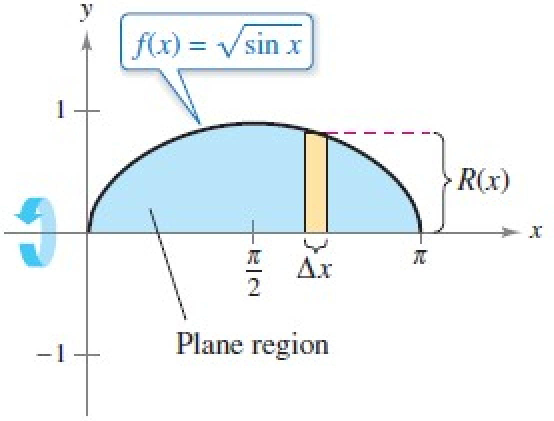
Figure 7.15



Vertical axis of revolution

Example 1 – *Using the Disk Method*

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the *x*-axis ($0 \le x \le \pi$) about the *x*-axis.



Solution:

From the representative rectangle in the upper graph in Figure 7.16, you can see that the radius of this solid is

$$R(x) = f(x)$$
$$= \sqrt{\sin x}.$$

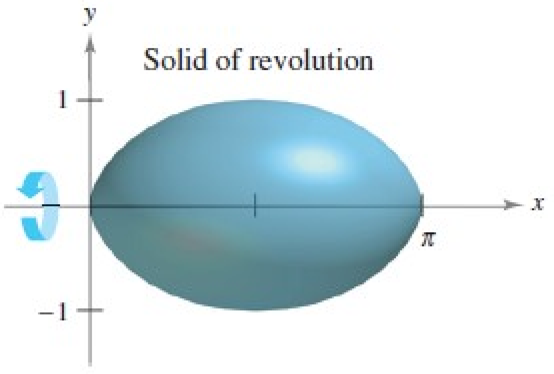


Figure 7.16

Example 1 – *Solution*

cont'd

So, the volume of the solid of revolution is

$$V = \pi \int_a^b [R(x)]^2 \, dx \qquad \text{Apply disk method.}$$

$$= \pi \int_0^\pi \left(\sqrt{\sin x}\right)^2 dx \qquad \text{Substitute } \sqrt{\sin x} \text{ for } R(x).$$

$$= \pi \int_0^\pi \sin x \, dx \qquad \text{Simplify.}$$

$$= \pi \Big[-\cos x\Big]_0^\pi \qquad \text{Integrate.}$$

$$= \pi(1 + 1)$$

$$= 2\pi.$$

The Washer Method

The Washer Method

The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative **washer.**

The washer is formed by revolving a rectangle about an axis, as shown in Figure 7.18.

If r and R are the inner and outer radii of the washer and w is the width of the washer, then the volume is given by

Volume of washer = $\pi(R^2 - r^2)w$.

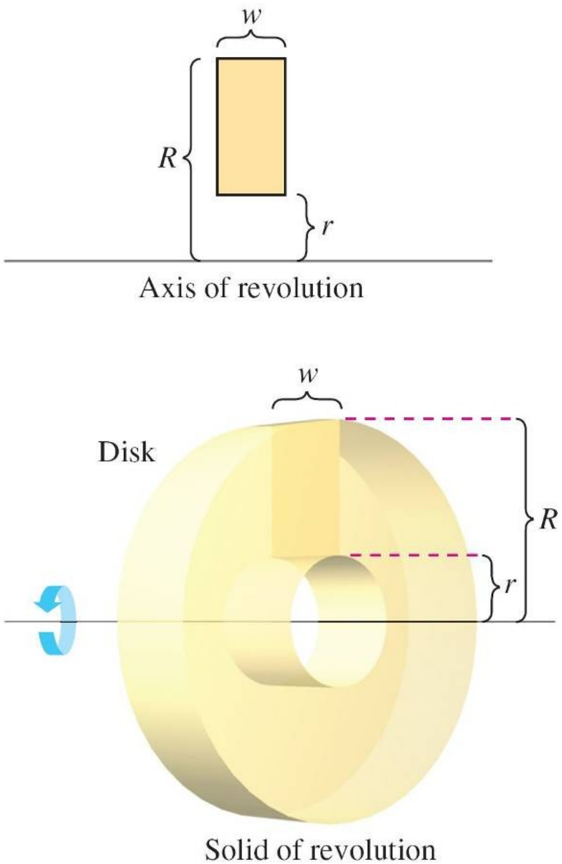


Figure 7.18

The Washer Method

To see how this concept can be used to find the volume of a solid of revolution, consider a region bounded by an **outer radius** $R(x)$ and an **inner radius** $r(x)$, as shown in Figure 7.19.

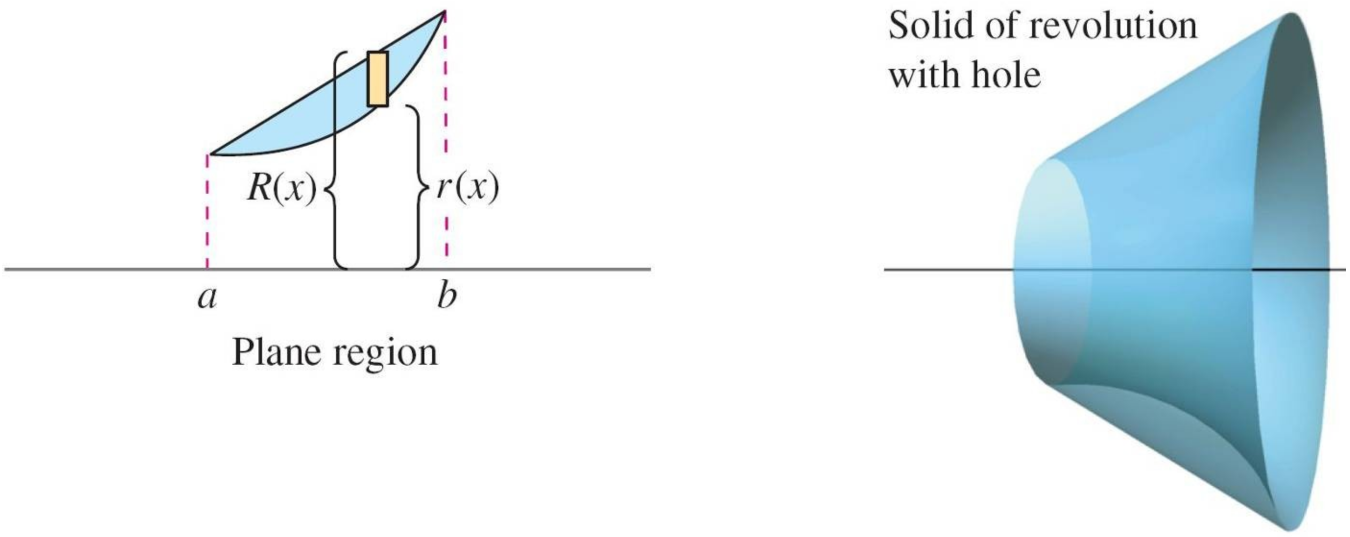


Figure 7.19

The Washer Method

If the region is revolved about its axis of revolution, the volume of the resulting solid is given by

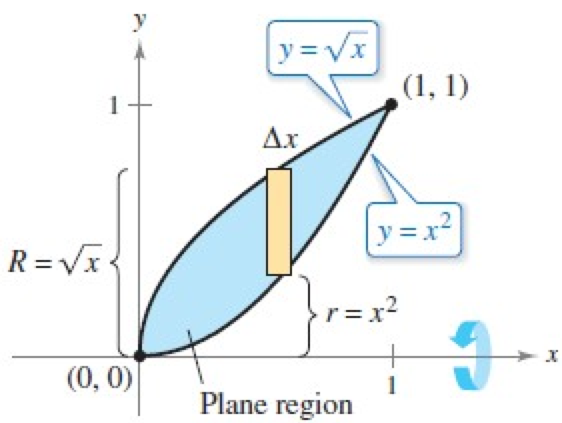
$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx. \qquad \text{Washer method}$$

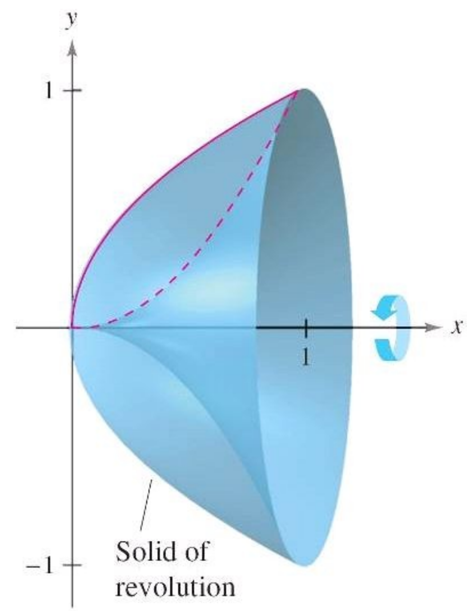
Note that the integral involving the inner radius represents the volume of the hole and is *subtracted* from the integral involving the outer radius.

Example 3 – *Using the Washer Method*

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the

x-axis, as shown in Figure 7.20.





Solid of revolution

Figure 7.20

Example 3 – *Solution*

In Figure 7.20, you can see that the outer and inner radii are as follows.

$$R(x) = \sqrt{x}$$ Outer radius

$$r(x) = x^2$$ Inner radius

Integrating between 0 and 1 produces

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2)\, dx$$ Apply washer method.

$$= \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2]\, dx$$ Substitute $\sqrt{x}$ for $R(x)$ and x^2 for $r(x)$.

Example 3 – *Solution*

cont'd

$$= \pi \int_0^1 (x - x^4)\, dx \qquad \text{Simplify.}$$

$$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 \qquad \text{Integrate.}$$

$$= \frac{3\pi}{10}.$$

The Washer Method

In each example so far, the axis of revolution has been *horizontal* and you have integrated with respect to x. In Example 4, the axis of revolution is *vertical* and you integrate with respect to y. In this example, you need two separate integrals to compute the volume.

Example 4 – *Integrating with Respect to y, Two-Integral Case*

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about y-axis, as shown in Figure 7.21.

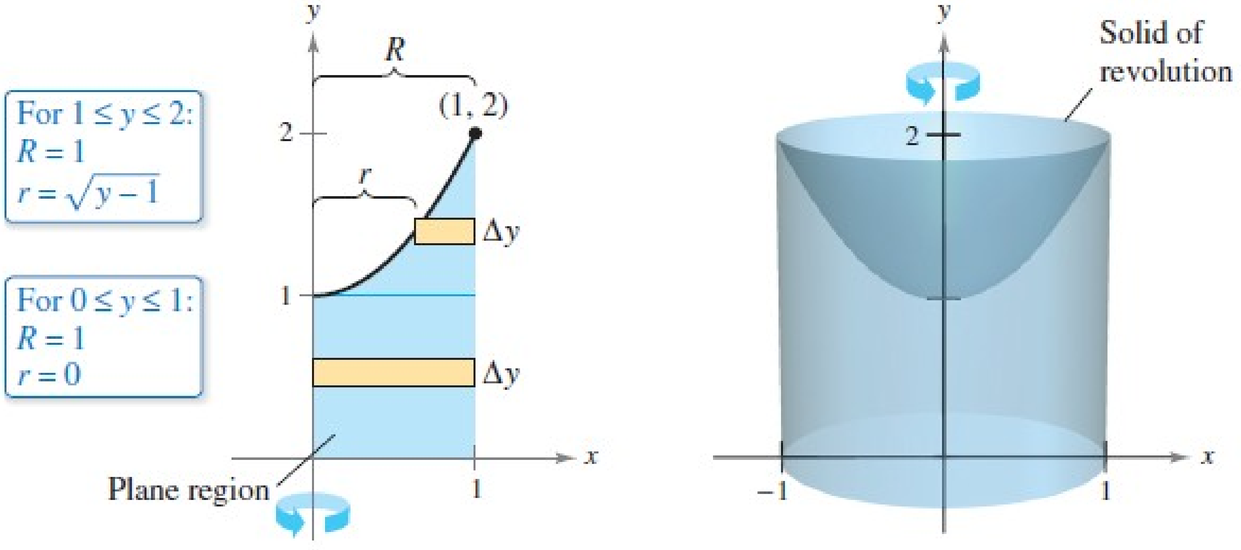


Figure 7.21

Example 4 – *Solution*

For the region shown in Figure 7.21, the outer radius is simply $R = 1$.

There is, however, no convenient formula that represents the inner radius.

When $0 \leq y \leq 1$, $r = 0$, but when $1 \leq y \leq 2$, r is determined by the equation $y = x^2 + 1$, which implies that $r = \sqrt{y - 1}$.

$$r(y) = \begin{cases} 0, & 0 \leq y \leq 1 \\ \sqrt{y - 1}, & 1 \leq y \leq 2 \end{cases}$$

Example 4 – *Solution*

cont'd

Using this definition of the inner radius, you can use two integrals to find the volume.

$$V = \pi \int_0^1 (1^2 - 0^2)\,dy + \pi \int_1^2 \left[1^2 - \left(\sqrt{y-1}\right)^2\right] dy$$ Apply washer method.

$$= \pi \int_0^1 1\,dy + \pi \int_1^2 (2 - y)\,dy$$ Simplify.

$$= \pi \Big[y \Big]_0^1 + \pi \left[2y - \frac{y^2}{2} \right]_1^2$$ Integrate.

$$= \pi + \pi \left(4 - 2 - 2 + \frac{1}{2} \right)$$

$$= \frac{3\pi}{2}$$

Example 4 – *Solution*

cont'd

Note that the first integral $\pi \int_0^1 1\, dy$ represents the volume of a right circular cylinder of radius 1 and height 1.

This portion of the volume could have been determined without using calculus.

Solids with Known Cross Sections

Solids with Known Cross Sections

With the disk method, you can find the volume of a solid having a circular cross section whose area is $A = \pi R^2$.

This method can be generalized to solids of any shape, as long as you know a formula for the area of an arbitrary cross section.

Some common cross sections are squares, rectangles, triangles, semicircles, and trapezoids.

Solids with Known Cross Sections

VOLUMES OF SOLIDS WITH KNOWN CROSS SECTIONS

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis,

$$\text{Volume} = \int_a^b A(x)\, dx.$$

See Figure 7.24(a).

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis,

$$\text{Volume} = \int_c^d A(y)\, dy.$$

See Figure 7.24(b).

(a) Cross sections perpendicular to x-axis

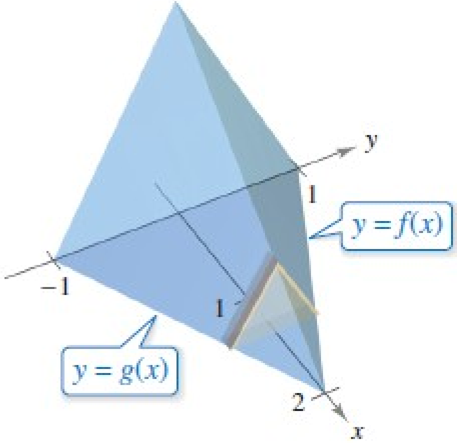
(b) Cross sections perpendicular to y-axis

Figure 7.24

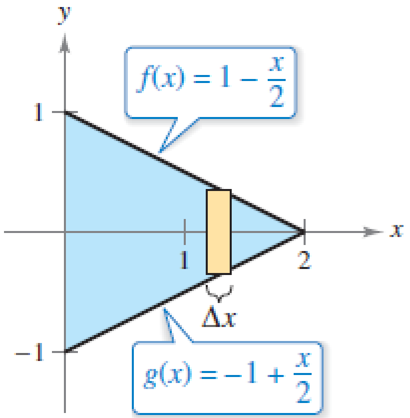
Example 6 – *Triangular Cross Sections*

Find the volume of the solid shown in Figure 7.25.

The base of the solid is the region bounded by the lines $f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$, and $x = 0$.



Cross sections are equilateral triangles.



Triangular base in *xy*-plane

Figure 7.25

The cross sections perpendicular to the *x*-axis are equilateral triangles.

Example 6 – *Solution*

The base and area of each triangular cross section are as follows.

$$\text{Base} = \left(1 - \frac{x}{2}\right) - \left(-1 + \frac{x}{2}\right) = 2 - x$$ Length of base

$$\text{Area} = \frac{\sqrt{3}}{4}(\text{base})^2$$ Area of equilateral triangle

$$A(x) = \frac{\sqrt{3}}{4}(2 - x)^2$$ Area of cross section

Example 6 – *Solution*

cont'd

Because *x* ranges from 0 to 2, the volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_0^2 \frac{\sqrt{3}}{4}(2 - x)^2\,dx$$

$$= -\frac{\sqrt{3}}{4}\left[\frac{(2 - x)^3}{3}\right]_0^2$$

$$= \frac{2\sqrt{3}}{3}.$$