# 7 Applications of Integration

# 7.1 Area of a Region Between Two Curves

# Objectives

- Find the area of a region between two curves using integration.
- Find the area of a region between intersecting curves using integration.
- Describe integration as an accumulation process.

# Area of a Region Between Two Curves

# Area of a Region Between Two Curves

With a few modifications, you can extend the application of definite integrals from the area of a region *under* a curve to the area of a region *between* two curves.

Consider two functions *f* and *g* that are continuous on the interval [*a*, *b*].

Also, the graphs of both *f* and *g* lie above the *x*-axis, and the graph of *g* lies below the graph of *f*, as shown in Figure 7.1.

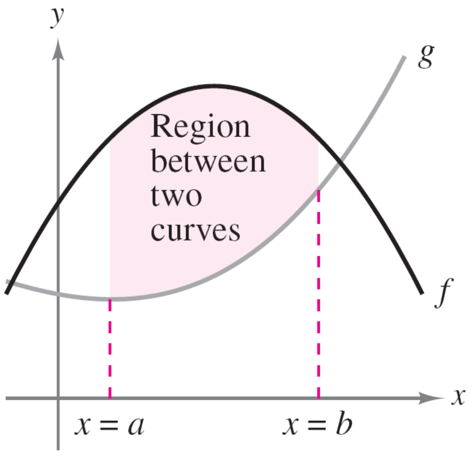
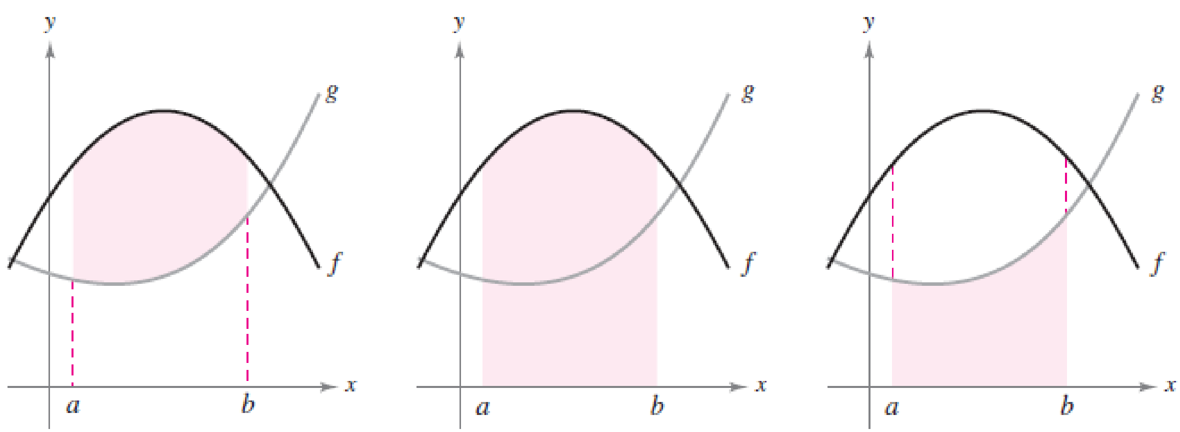



**Figure 7.1**

# Area of a Region Between Two Curves

You can geometrically interpret the area of the region between the graphs as the area of the region under the graph of *g* subtracted from the area of the region under the graph of *f*, as shown in Figure 7.2.




| Area of region between $f$ and $g$ | = | Area of region under $f$ | − | Area of region under $g$ |
|---|---|---|---|---|

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

**Figure 7.2**

# Area of a Region Between Two Curves

To verify the reasonableness of the result shown in Figure 7.2, you can partition the interval [*a*, *b*] into *n* subintervals, each of width $\Delta x$.

Then, as shown in Figure 7.3, sketch a **representative rectangle** of width $\Delta x$ and height $f(x_i) - g(x_i)$, where $x_i$ is in the $i$th subinterval.

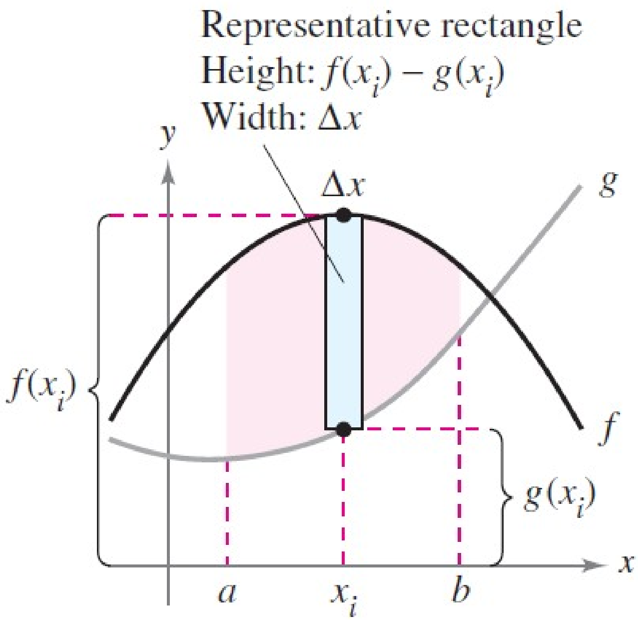



**Figure 7.3**

# Area of a Region Between Two Curves

The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x.$$

By adding the areas of the $n$ rectangles and taking the limit as $\|\Delta\| \to 0$ ($n \to \infty$), you obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\,\Delta x.$$

Because $f$ and $g$ are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. So, the area of the given region is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\,\Delta x$$

$$= \int_a^b [f(x) - g(x)]\,dx.$$

# Area of a Region Between Two Curves

**Area of a Region Between Two Curves**

If $f$ and $g$ are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all $x$ in $[a, b]$, then the area of the region bounded by the graphs of $f$ and $g$ and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

# Area of a Region Between Two Curves

In Figure 7.1, the graphs of *f* and *g* are shown above the *x*-axis. This, however, is not necessary.

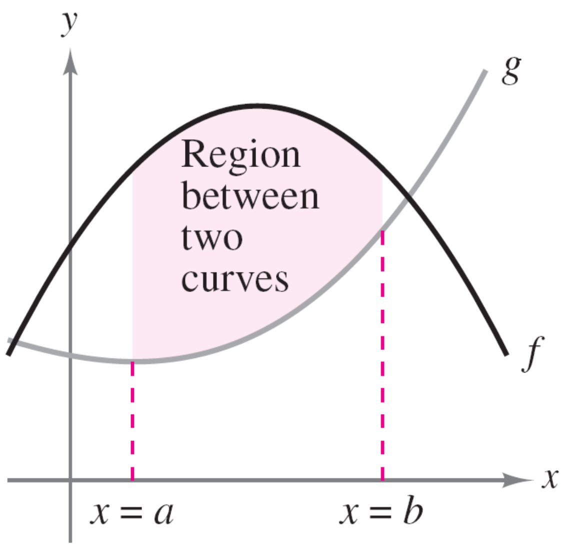



**Figure 7.1**

The same integrand $[f(x) - g(x)]$ can be used as long as *f* and *g* are continuous and $g(x) \le f(x)$ for all *x* in the interval $[a, b]$.

# Area of a Region Between Two Curves

This is summarized graphically in Figure 7.4.

Notice in Figure 7.4 that the height of a representative rectangle is $f(x) - g(x)$ regardless of the relative position of the $x$-axis.

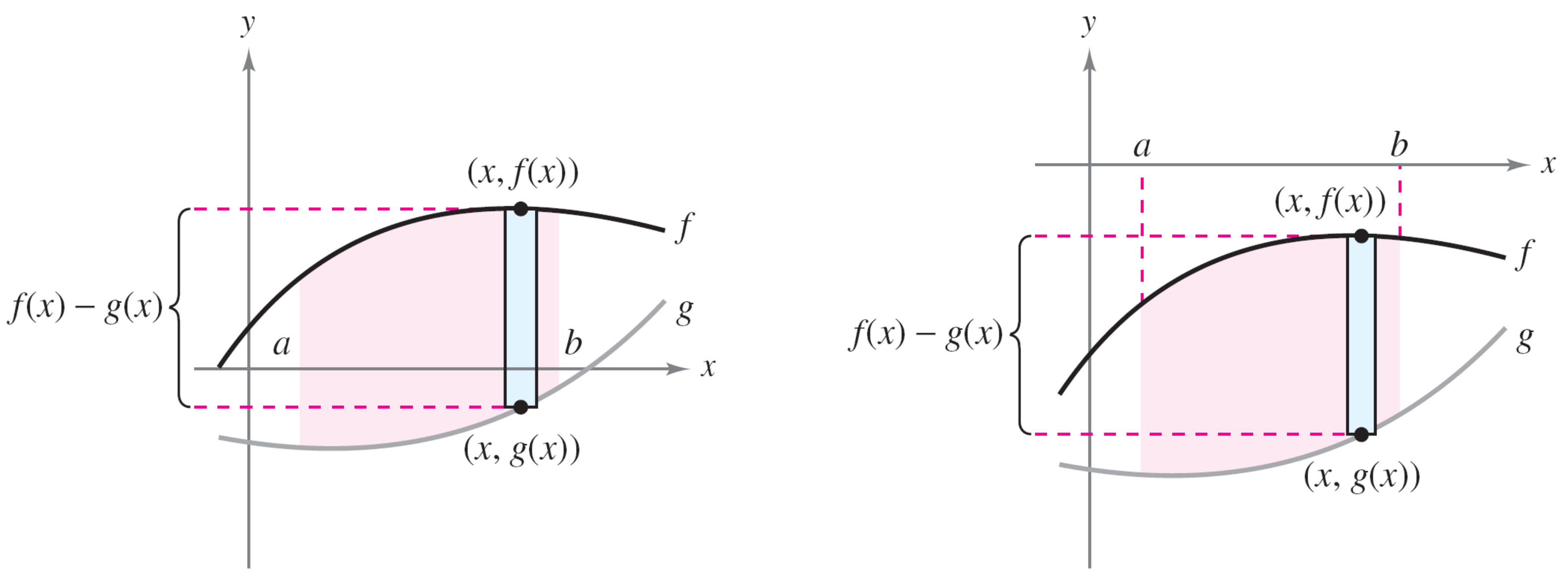



**Figure 7.4**

# Area of a Region Between Two Curves

Representative rectangles are used throughout this chapter in various applications of integration.

A vertical rectangle (of width $\Delta x$) implies integration with respect to *x*, whereas a horizontal rectangle (of width $\Delta y$) implies integration with respect to *y*.

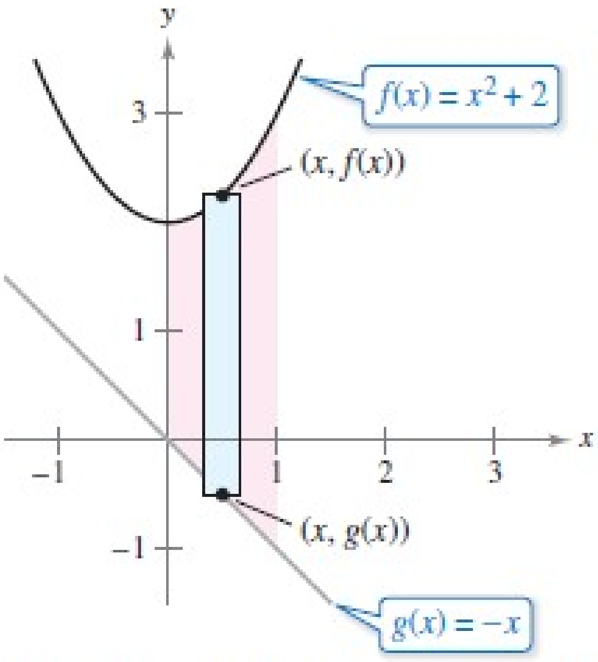
# Example 1 – *Finding the Area of a Region Between Two Curves*

Find the area of the region bounded by the graphs of $f(x) = x^2 + 2$, $g(x) = -x$, $x = 0$, and $x = 1$.

Solution:

Let $g(x) = -x$ and $f(x) = x^2 + 2$.

Then $g(x) \leq f(x)$ for all $x$ in $[0, 1]$, as shown in Figure 7.5.




Region bounded by the graph of $f$, the graph of $g$, $x = 0$, and $x = 1$

Figure 7.5

# Example 1 – *Solution*

cont'd

So, the area of the representative rectangle is

$$\Delta A = [f(x) - g(x)]\Delta x$$

$$= [(x^2 + 2) - (-x)]\Delta x$$

and the area of the region is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

$$= \int_0^1 [(x^2 + 2) - (-x)]\,dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{2} + 2$$

$$= \frac{17}{6}.$$

# Area of a Region Between Intersecting Curves

# Area of a Region Between Intersecting Curves

In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and the values of $a$ and $b$ are given explicitly.

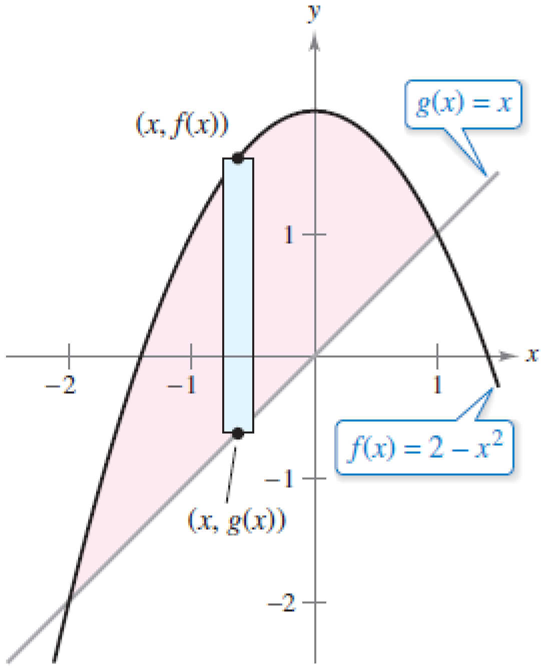
A more common problem involves the area of a region bounded by two *intersecting* graphs, where the values of $a$ and $b$ must be calculated.

# Example 2 – *A Region Lying Between Two Intersecting Graphs*

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

Solution:

In Figure 7.6, notice that the graphs of $f$ and $g$ have two points of intersection.




Region bounded by the graph of $f$ and the graph of $g$



**Figure 7.6**

# Example 2 – *Solution*

cont'd

To find the $x$-coordinates of these points, set $f(x)$ and $g(x)$ equal to each other and solve for $x$.

| | |
|---|---|
| $2 - x^2 = x$ | Set $f(x)$ equal to $g(x)$ |
| $-x^2 - x + 2 = 0$ | Write in general form. |
| $-(x + 2)(x - 1) = 0$ | Factor |
| $x = -2$ or $1$ | Solve for $x$. |

So, $a = -2$ and $b = 1$.

# Example 2 – *Solution*

cont'd

Because $g(x) \leq f(x)$ for all $x$ in the interval [–2, 1], the representative rectangle has an area of

$$\Delta A = [f(x) - g(x)]\Delta x$$

$$= [(2 - x^2) - x]\Delta x$$

and the area of the region is

$$A = \int_{-2}^{1} [(2 - x^2) - x]\, dx$$

$$= \left[ -\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^{1}$$

$$= \frac{9}{2}.$$

# Integration as an Accumulation Process

# Integration as an Accumulation Process

The integration formula for the area between two curves was developed by using a rectangle as the *representative element*.

For each new application in the remaining sections of this chapter, an appropriate element will be constructed using precalculus formulas you already know.

Each integration formula will then be obtained by summing or accumulating these representative elements.

Known precalculus formula ⇒ Representative element ⇒ New integration formula

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# Integration as an Accumulation Process

For example, the area formula in this section was developed as follows.

$$A = (\text{height})(\text{width}) \quad \Rightarrow \quad \Delta A = [f(x) - g(x)]\,\Delta x \quad \Rightarrow \quad A = \int_a^b [f(x) - g(x)]\,dx$$

## Example 6 – *Describing Integration as an Accumulation Process*

Find the area of the region bounded by the graph of $y = 4 - x^2$ and the $x$-axis. Describe the integration as an accumulation process.

Solution:

The area of the region is given by

$$A = \int_{-2}^{2} (4 - x^2)\, dx.$$

You can think of the integration as an accumulation of the areas of the rectangles formed as the representative rectangle slides from $x = -2$ to $x = 2$, as shown in Figure 7.11.

# Example 6 – *Solution*

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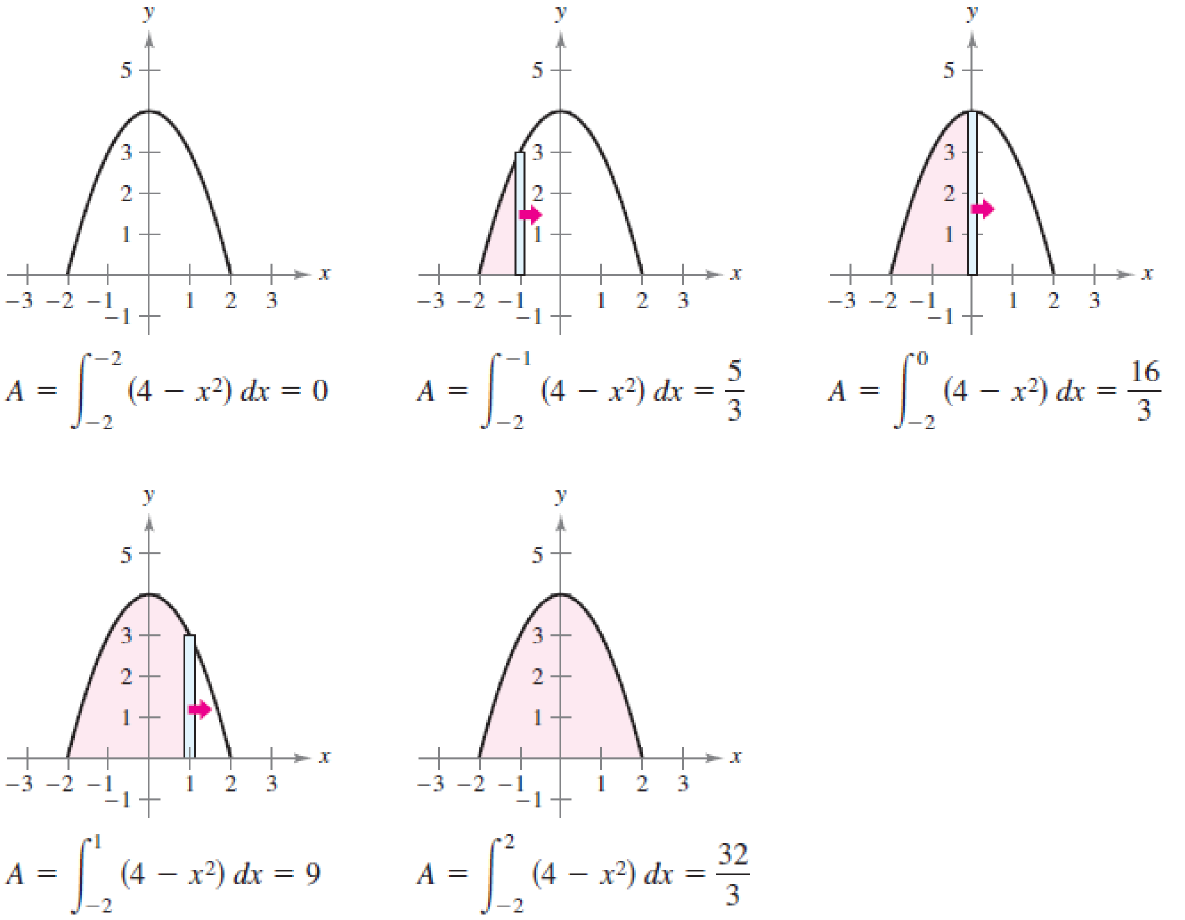


**Figure 7.11**