Modeling using ODEs: Population Model

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Growth Processes

- The U.S. Bureau of the Census frequently predicts population trends.
  To do this, demographers use known population data to formulate "laws" of population change.
- Each law is converted into a mathematical formula that can then be used to make the predictions. Since there is a good deal of uncertainty about which law best describes actual population shifts, several are derived, each proceeding from different assumptions.
- Similarly, ecologists make predictions about changes in the numbers of fish and animal populations, and biologists formulate laws for changing densities of bacteria growing in cultures or even growth of tumors.

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Goals

- Introduce general laws of population change.
- Solve the corresponding mathematical models.
- Interpret the results in terms of the growth, stabilization, or decline of a population.

Growth Models

Denote the population of a species at time t by $y(t)$.

- The values of y(t) are non negative integers and change by integer amounts as time goes on.
- For a large population an increase by one or two over a short time span is infinitesimal relative to the total, and we may think of the population as changing continuously instead of by discrete jumps.
- Assume that y(t) is continuous, we might as well smooth off any corners on the graph of y(t) and assume that the function is differentiable.
- If we had let y(t) denote the population density (i.e., the number per unit area or volume of habitat), the continuity and differentiability of y(t) would have seemed more natural. However, we shall continue to interpret y(t) as the size of the population, rather than density.

Underlying Principle

$$\text{Net Change} = \text{Rate In - Rate Out}$$

where

- Rate In: Sum of birth and immigration rates.
- Rate Out: Sum of the death and the emigration rates.

$$\text{Net Change} = \underbrace{\textit{Birth} - \textit{Death}}_{\text{internal rate } Ry(t)} - \underbrace{\text{Immigration-Emigration}}_{\text{external rate } M}$$

The study of population changes to the problem of solving the following ODE

$$y'(t) = Ry(t) + M, \;\; y(0) = y_0.$$

Exponential Growth

Malthus, a professor of history and political economy claimed that It may safely be pronounced, therefore, that population, when unchecked, goes on doubling itself every twenty-five years. The Malthusian principle of explosive growth of human populations has become one of the classic laws of population change.

Let $M = 0$ and $R = r$ a positive constant, then, the population indeed grows exponentially $y(t) = y_0 e^{rt}$ for all $t \geq 0$.

How long does it take for the population to double itself?

Answer:

$$\boxed{T = \frac{ln2}{r}, \;\; r = 0.02777.}$$

Logistic Growth

The unbridled growth of a population (as predicted by the simple Malthusian law of exponential increase) cannot continue forever.
Malthus claimed that resources grow atmost arithmetically, i.e., the net increase in resources each year does not exceed a fixed constant.
An exponential increase in the size of a population must soon outstrip available resources.
The resulting hardships would surely put a damper on growth.

Logistic Growth

Let

- $y(t)$: denote the population of a species at time t.
- r: Logistic Growth rate
- K: Carrying Capacity of the species.

A more realistic model is

$$y'(t) = r(1 - \frac{y(t)}{K})y(t), \ \ y(0) = y_0.$$

The analytical solution is:

$$y(t) = \frac{Ky_0}{y_0 + \left(K - y_0\right)e^{-rt}}.$$

Gompertz Equation

Let

- $y(t)$: denote the population of tumor cells at time t.
- r: Intrinsic Growth rate of cells
- K: Carrying Capacity i.e., maximum size that the population can attain with the available nutrients.

$$y'(t) = -r \ln\left(\frac{y(t)}{K}\right) y(t), \;\; y(0) = y_0.$$

The analytical solution is:

$$y(t) = \frac{K}{(K/y_0)^{e^{-rt}}}.$$

General Single Species Population Model

Let

- a : be a positive parameter characterizing the population model.
- $y(t)$: denote the population of a species at time t.
- r: Logistic Growth rate
- K: Carrying Capacity of the species.

$$y'(t) = \frac{r}{a}\left(1 - \left(\frac{y(t)}{K}\right)^a\right)y(t), \;\; y(0) = y_0.$$

The analytical solution is:

$$y(t) = \frac{Ky_0}{\left(y_0^a + \left(K^a - y_0^a\right)e^{-rt}\right)^{1/a}}.$$