Modeling using ODEs: Mixing Tank Problem

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Reference

W. Boyce and R. DiPrima, Elementary Differential Equations and Boundary Value Problems 8th Edition, John Wiley and Sons, 2005.

First-Order Ordinary Differential Equation

Definition (First Order Ordinary Differential Equation)

Let $\Omega \subset \mathbb{R}^2$, and let $F : \Omega \to \mathbb{R}$ be a continuous function. Then,

$$\frac{du(t)}{dt} = F(t, u(t)) \tag{1}$$

is a first-order Ordinary Differential Equation (ODE) in unknown function $u(t)$.
A function $y : [a, b] \to \mathbb{R}$ is called a solution of the ODE (1) provided it satisfies the equation (1) at each $t \in [a, b]$.

Initial Value Problem

Definition (Initial Value Problem)

Let $\Omega \subset \mathbb{R}^2$, and let $F : \Omega \to \mathbb{R}$ be a continuous function and $u_0 \in \mathbb{R}$. Then,

$$\frac{du(t)}{dt} = F(t, u(t)) \tag{2}$$

$$u(0) = u_0, \tag{3}$$

is an initial value problem (IVP) for the first-order ODE (1).

A function $y : [a, b] \to \mathbb{R}$ is called a solution of the IVP (2)–(3) provided it satisfies the equation (2) and (3) at each $t \in [a, b]$.

First-Order Ordinary Differential Equation

Notation: We assume the following shorthand:

$$u \equiv u(t)$$

so that (1) assumes the form

$$\frac{du}{dt} = F(t, u) \tag{4}$$

$$\frac{du}{dt} = 10u \tag{5}$$

has a solution $u(t) = e^{10t}$.
Such solutions are called closed form solutions since they can be expressed in terms of a formula.

Separable ODE

For specific forms of the right hand side function $F(t, u)$ of our ODE (4), we have a closed form solution.

Definition (Separable ODE)

The following ODE is said to be separable

$$\frac{du}{dt} = F(t, u)$$

if $F(t, u) = f(t)g(u)$ where f and g are integrable functions.

The closed form solution is given by:

$$\int \frac{du}{dt} = \int F(t, u) = \int f(t)g(u)$$

$$\int \frac{1}{g(u)} du = \int f(t)\ dt$$

Some more definitions

Definition (Autonomous ODE)

An ODE (1) is said to be autonomous if the right hand side function $F(t,u)$ is a function of u and does not depend on t explicitly.

Definition (Linear ODE)

An ODE (1) is said to be linear if u appears linearly in the right hand side function $F(t,u)$.

1. $\frac{du}{dt} = \sin(u)$
2. $\frac{du}{dt} = ue^t$
3. $\frac{du}{dt} = e^u t$

Mixing Tank Separable ODEs

When studying separable differential equations, one classic class of examples is the mixing tank problems. Here we will consider a few variations on the following benchmark example.

Benchmark Example

Example

A tank has pure water flowing into it at 10 l/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Initially, the tank contains 10 kg of salt in 100 l of water.
How much salt will there be in the tank after 30 minutes?

Approach

To study such a question, we consider the rate of change of the amount of salt in the tank.

- Let S be the amount of salt in the tank at any time t.
- If we can create an equation relating $\frac{dS}{dt}$ to S, then we will have a differential equation which we can solve to determine the relationship between S and t.
- To describe $\frac{dS}{dt}$, we use the concept of concentration, the amount of salt per unit of volume of liquid in the tank.

Approach

- In this example, the inflow and outflow rates are the same, so the volume of liquid in the tank stays constant at 100 l. Hence, we can describe the concentration of salt in the tank by

$$\text{concentration of salt} = \frac{S}{100}$$

in kg/l.

- Then, since mixture leaves the tank at the rate of 10 l/min, salt is leaving the tank at the rate of

$$\frac{S}{100}10 = S/10$$

in l/min.

- This is the rate at which salt leaves the tank, so

$$\frac{dS}{dt} = -\frac{S}{10}$$

Approach

- $$S(t) = Ce^{-t/10}$$

  solves the differential equation with C is a constant which can be determined by using the initial condition:

  $$S(0) = 10$$

  which yields $C = 10$.
- Thus, the amount of salt in the mathematical model is given by
  $$S(t) = 10e^{-t/10}$$
- The amount of salt in the tank after 30 minutes is 0.5 kg.
- The amount of salt will go to zero with the passage of time.

Tweaking the benchmark problem

Example

A tank has pure water flowing into it at 10 l/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Salt is added to the tank at the rate of 0.1 kg/min.
Initially, the tank contains 10 kg of salt in 100 l of water. How much salt will there be in the tank after 30 minutes?

How does the ODE change?

$$\frac{dS}{dt} = -\frac{S}{10} + 0.1.$$

The solution is $S(t) = 1 + 9e^{-0.1t}$.
The amount of salt in the tank after 30 minutes is 1.448 kg.

Another example

Example

A tank has pure water flowing into it at 12 l/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Initially, the tank contains 10 kg of salt in 100 l of water.

Since the inflow rate is greater than the outflow rate, the volume is not constant and the volume grows as

$$V(t) = 100 + 2t.$$

Thus, the concentration of salt after t minutes is

$$\frac{S}{V} = \frac{S}{100 + 2t}.$$

Another example

$$\frac{dS}{dt} = -\frac{S}{100+2t}(10)$$

Answer:

$$S(t) = \frac{10^{11}}{(100+2t)^5}.$$

After 30 minutes, there will be 0.95 kg of salt in the tank. Why is this more than benchmark example?