Math 4370 | **COMBINATORICS** | Spring 2012

Dr. Duval | **Homework**

Thursday, April 26

Follow the separate general guidelines for Parts A,B,C. Be sure to include and label *all four* standard parts (a), (b), (c), (d) of Part A in what you hand in.

Counting trees on labeled vertices, part 1

Section 5.3, up through and including Example 5.12

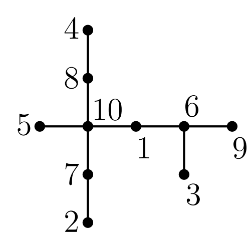
A: Reading questions. Due by 3pm, Mon., 30 Apr.

1. Even though this section is about **labeled** trees, it will be helpful to start with this simple problem about **unlabeled** trees: List all different unlabeled trees on 4 vertices.
2. How many labeled trees on 4 vertices are there? List them all.
3. Most of this reading is devoted to the proof of Theorem 5.10, probably the most involved proof we have seen this semester. So it is probably too much to ask you to understand the complete proof before class, but it will still be worthwhile to start working on the proof. Why is Equation (5.1) equivalent to Theorem 5.10?
4. What is a **doubly-rooted tree**? Draw an example of a doubly-rooted tree on 7 vertices. Why are doubly-rooted trees counted by the left-hand side of Equation (5.1)?
5. What is the **short diagram** of a function from $[n]$ into $[n]$? Give an example when $n = 7$.
6. The heart of the proof of Theorem 5.10 is a bijection between the set A of functions from $[n]$ into $[n]$, and the set B of doubly-rooted trees on $[n]$. Try to follow this bijection with the Example 5.11: Fill in the details that show why the function f given in Example 5.11 corresponds to the graph shown in Figure 5.13 in this bijection. In other words, try to follow along the proof of Theorem 5.10, starting from the second paragraph on p. 267, and interpret each sentence with this example.

B: Warmup exercises. For you to present in class. Due by the end of class Tue., 1 May.

1. Illustrate your understanding of the proof of Cayley's theorem by:
   (a) Finding the doubly-rooted tree corresponding to the function found below; and
   (b) Finding the function $[n] \to [n]$ corresponding to the doubly-rooted tree found below, with start vertex 4 and end vertex 9.

i	$f(i)$
1	6
2	10
3	1
4	10
5	6
6	10
7	2
8	6
9	2
10	3



Counting trees on labeled vertices, part 2
Section 5.3, pp. 269–274

A: Reading questions. Due by 3pm, Wed., 2 May.

1. List all the labeled trees on 3 vertices (use labels 1, 2, and 3). For each tree, also write down its degree sequence. For each degree sequence, write down the monomial in variables x_1, x_2, and x_3, as described in pp. 269–270. Finally add up these monomials to get the generating function (a polynomial in 3 variables) for trees with 3 vertices described in equation (5.2). Can you factor this polynomial in any interesting way?
2. Repeat the previous reading question for trees with 4 vertices. (This is a larger enterprise, so you'll need to be systematic, and take advantage of symmetry.)
3. Can you explain why the generating function for F_n always has a factor of $x_1 x_2 \ldots x_n$?
4. Explain how Theorem 5.13 implies Theorem 5.10. (The proof of Theorem 5.13 is probably too hard to read and understand on your own, but we will spend the bulk of class time working through this proof.)
5. What will you do with all the time you have, now that there are no more reading questions to answer?

B: Warmup exercises. For you to present in class. Due by the end of class Thu., 3 May.

1. Nothing to prepare in advance, but be ready to work through examples that illustrate steps of the proof of Theorem 5.13.