2 Homework 2

1. (a) (not to turn in) Review how counting dots can be used to explain why the addition of positive integers is commutative.

   (b) Using a parallel structure to the above, explain why, for positive integer powers, the product of powers rule holds: for all $a \in \mathbb{R}, x_1, x_2 \in \mathbb{Z}_{>0}$, we have $a^{x_1}a^{x_2} = a^{x_1+x_2}$. Be sure to use the definition of positive integer power.

2. In the following explanations, be sure to:

   - Use the definition of a^{-1}
   - Use the definition of a^{-n}, where $n \in \mathbb{N}$

   (a) (not to turn in) Review our in-class work on *Natural and Integer Powers: Do Power Properties Hold?*

   (b) (not to turn in) Review how a movement and location model can be used to explain why the product of two negative integers is positive.

   (c) Using a parallel structure to the above, explain why the power of power rule holds for negative integer powers of nonzero bases: for all $a \in \mathbb{R}, x_1, x_2 \in \mathbb{Z}$, with $a \neq 0$ and $x_1, x_2 < 0$, we have $(a^{x_1})^{x_2} = a^{x_1 x_2}$.

3. In the following explanations, be sure to:

   - Use the definition of $a^{1/q}$
   - Use the definition of $a^{p/q} = (a^{1/q})^p$
   - Use the definitions of a^{-1} and a^{-n}, for $n \in \mathbb{N}$

   (a) Explain why for all $a \in \mathbb{R}, a > 0$, and $p, q \in \mathbb{N}_{>0}$, we have $(a^p)^{-\frac{1}{q}} = (a^{-\frac{1}{q}})^p$.

   (b) Explain why for all $a \in \mathbb{R}, a > 0$, and $p, q \in \mathbb{N}_{>0}$, we have $(a^{-p})^{-\frac{1}{q}} = (a^{-\frac{1}{q}})^{-p}$.

4. Use graphing technology to explore each of the following functions and then make a conjecture about the values of $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$. (You do not need to prove your conjecture.)

   (a) $f(x) = (2^{-\frac{1}{x^2}})^{-x}$

   (b) $f(x) = (a^{-\frac{1}{x}})^x$

   Note –

   Theorem (0^0 is an indeterminate form)**.** For every real number L, there are functions $B : \mathbb{R} \to \mathbb{R}, E : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to 0^+} B(x) = 0$, $\lim_{x\to 0^+} E(x) = 0$, and $\lim_{x\to 0^+} B(x)^{E(x)} = L$.

   Note on theorem. We do not address the proof here, though the above task gives an indication of how wacky the behavior of limits to "0^0" can be. Because of the infinitely many possibilities for limits to "0^0", this expression is called an "indeterminate form". It is a special and particularly pathological case of how a limit can be undefined. □

5. (a) Which of the following equations have solutions? Explain why each does or does not have a solution.

   $$5^x = 25 \qquad (6.7)^x = 25 \qquad (\tfrac{1}{10})^x = 25 \qquad (1.001)^x = 25 \qquad 1^x = 25 \qquad 0^x = 25$$

   (b) Let $a \in \mathbb{R}, a > 0$. Suppose $a^x = a^\pi$. True or false? $x = \pi$.