1 Homework 1

Problems

1. In *Reference: Highlights of Number and Operation, K-8*, two diagrams of counting dots are given as a way to visualize the commutative property of addition and multiplication for natural numbers. Write some sentences that go with these diagrams to explain what is going on and why it is reasonable to conclude that the addition and multiplication of *any* two natural numbers commute, not just 5 and 3. (Limit your explanation to 5 lines.)

2. (a) (not to turn in) Review the explanation in Chapter 1 for the definition of movement by -1, and make sure that you understand the explanation.

   "-1 is defined as the additive inverse of 1, meaning $-1+1=0$. So moving -1 after moving 1 should result in zero net movement. So moving -1 should be moving the same distance as 1, but in the opposite direction."

   (b) Using language as parallel as possible to the explanation of (a), justify why:

   - Defining a^{-1} as the multiplicative inverse of a makes sense. Use the Power Properties and the definition of multiplicative inverse.
   - Defining a^{-n} as the multiplicative inverse of $\frac{1}{a^n}$ (the multiplicative inverse of a^n) makes sense. Use the Power Properties and the definition of multiplicative inverse.

   (c) Using the model of movement and multiplication, explain why $-n$ (the additive inverse of n) is equal to $(-1)\times n$.

   (d) Using language as parallel as possible to the explanation in (c), justify why it makes sense that $\frac{1}{a^n}$ should represent the same quantity as $\left(\frac{1}{a}\right)^n$ (the n-th power of the multiplicative inverse of a).

3. Using the models for movement and multiplication described in Chapter 1, explain why:

   (a) $(-1)\times(-1)=1$

   (b) $(-3)\times(-2)=6$. (Limit your explanation to 5 lines.)

   (c) $-5+3=3+(-5)$ The sides of this equation mean: Start at location -5, move $+3$; Start at 3, move -5. (Hint: Count the number of movements $+1$ compared to the movements of -1, and use the fact that movements of -1 paired with movements of 1 result in a net movement of 0. Limit your explanation to 5 lines.)

4. For these, you can take as a given that multiplication is well-defined, meaning that when you multiply two real numbers, there is only one possible result of this. You may also take associativity of multiplication of reals as given, as well as commutativity of integers (but not fractions).

   (a) Explain why there is only one number that can be the result of $5\div 3$, using the definition of division. (Definition of division: We say $a\div b$ is x if $x\cdot b=a$.) (Hint: What happens if there were two numbers where one was smaller or larger than the other? What would happen when you try to take three copies of these numbers?)

   (b) Explain why there is only one number represented by $\frac{1}{3}$, using the definition of fraction.

   (c) Explain, based on the previous parts of this problem, why $5\div 3$ is the same quantity as $\frac{1}{3}\cdot 5$. (Hint: Show that multiplying one of them by a particular number must result in the other quantity.)

5. Read this blog post
   https://illustrativemathematics.blog/2018/05/14/untangling-fractions-ratios-and-quotients/,
   which references the example in Problem 4. What do you take away from this post about students learning number and operation?

6. In class, we discussed why students might struggle with understanding why $5\div 3$ represents the same quantity as $5\cdot\frac{1}{3}$. Building on this: Why might students struggle with understanding why $\sqrt[3]{a^5}$ and $(\sqrt[3]{a})^5$ represent the same quantity? Or in general, why $\sqrt[q]{a^p}$ and $(\sqrt[q]{a})^p$ represent the same quantity? (Limit your explanation to three lines.)

7. (a) Justify why it makes sense to define $a^{-5/3} = \frac{1}{\underbrace{(a^{1/3})(a^{1/3})(a^{1/3})(a^{1/3})(a^{1/3})}_{5}}$.

   Use the Power Properties and the definition of multiplicative inverse.

   (b) Justify why it makes sense, for $p < 0$, to define $a^{p/q} = \frac{1}{\underbrace{(a^{1/q}) \cdot (a^{1/q}) \cdots (a^{1/q})}_{|p|}}$. Assume $p, q \in \mathbb{Z}$.

   Use the Power Properties and the definition of multiplicative inverse.

8. (a) Use the definition of fraction and the number line to explain how to write $\frac{p}{q} + \frac{r}{s}$ as a single fraction.

   (b) Use the definition of fraction and the number line to explain how to write $\frac{p}{q} \cdot \frac{r}{s}$ as a single fraction.

   (c) Using language that is as parallel a possible to your explanations above, find and justify formulas for $a^{\frac{p}{q}+\frac{r}{s}}$ and $a^{\frac{p}{q}\cdot\frac{r}{s}}$.

9. Here are some places where limits and convergence can appear in high school curricula:

   - Geometry: *Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri?s principle, and informal limit arguments.* (Common Core G-GMD.1)
   - Pre-calculus: *Limits of functions and sequences.* (Lincoln Public School standards and course descriptions, https://home.lps.org/math/secondary/#)
   - AP Calculus: *The derivative of a function is defined as the limit of a difference quotient and can be determined using a variety of strategies. (College Board Course and Exam Description: AP Calculus AB and AP Calculus BC, https://apcentral.collegeboard.org/pdf/ap-calculus-ab-and-bc-course-and-exam-description.pdf)*

   (a) Choose a topic from the above. Describe an example of a convergent sequence that you might use while teaching one of geometry, pre-calc, or calculus.

   (b) What is an (informal) 1-2 sentence explanation of what it means for a sequence to converge or limit to a value?

10. (a)
   - Using Desmos, graph in one window:
     - $f(x) = 5^x$
     - $f_1(x) = 5.1^x$
     - $f_3(x) = 5.001^x$
     - $f_6(x) = 5.000001^x$.
   - Using the settings feature, change your window to display only the x-coordinates in the interval $[\pi - 1, \pi + 1]$, and center the y coordinates around $y = 5^\pi$.
   - Sketch what you see.
     - Label the point $(\pi, 5^\pi)$
     - Point to where f and each variation of f are.

   (b)
   - Clear the screen. Then graph in one window:
     - $z(x) = 0^x$
     - $z_1(x) = 0.1^x$
     - $z_3(x) = 0.001^x$
     - $z_6(x) = 0.000001^x$
   - Using the settings feature, change your window to display only the x-coordinates in the interval $[-1, 1]$, and center the y coordinates so you can see $y = 0$ and $y = 1$.
   - Sketch what you see.
     - Label the point $(0, 1)$ and $(0, 0)$
     - Point to where z and each variation of z are.

   (c) What do you think the slope of $\lim_{n\to\infty} \left(0 + \frac{1}{10^n}\right)^x$ would be, near $x = 0$? (Answer only)

   (d) How do you think the slope of $\lim_{n\to\infty} \left(5 + \frac{1}{10^n}\right)^x$ compares to the slope of 5^x, near $x = \pi$? (Answer only)

11. Read the statement of this theorem:

    Theorem. *For all positive $a \in \mathbb{R}$ and positive $q \in \mathbb{Z}$, there exists a unique positive q-th root of a.*

    The word "positive" is used three times, to refer to different things. Using the notation $a^{1/q}$, and the terms "exponent", "base", and "power", explain what each "positive" is talking about:

    - The first "positive" is talking about ________________________.
    - The second"positive" is talking about ________________________.
    - The third "positive" is talking about ________________________.

    Read over the proof of the Lemma below. Why does the Theorem help prove the lemma? How would you explain the logic of the proof?

    Lemma. *For all $a \in \mathbb{R}$, $p, q \in \mathbb{Z}, q \neq 0$, if $\sqrt[q]{a}$ and $\sqrt[q]{a^p}$ exist, then*

    $$\sqrt[q]{a^p} = (\sqrt[q]{a})^p.$$

    Proof of Lemma. Given $a \in \mathbb{R}$, $p, q \in \mathbb{Z}, q \neq 0$, and $\sqrt[q]{a}$ and $\sqrt[q]{a^p}$ exist.

    $$\begin{aligned} ((\sqrt[q]{a})^p)^q &= (\sqrt[q]{a})^{pq} && \text{(power of a power works for all natural powers)} \\ &= (\sqrt[q]{a})^{qp} && \text{(power of a power for natural powers)} \\ &= ((\sqrt[q]{a})^q)^p && \text{(power of a power for natural powers)} \\ &= a^p && \text{(definition of } q\text{-th root).} \end{aligned}$$

    We have shown $(\sqrt[q]{a})^p$ is a q-th root of a^p, so by the Theorem, $\sqrt[q]{a^p} = (\sqrt[q]{a})^p$. □

    Explanation of why the Theorem helps make the proof work:

12. As a follow-up to the Double Sunglasses problem, let's look again at the (incorrect) potential answer 100%, of what is the equivalent tint of double 50% sunglasses.

    Suppose a student says 100%.

    Why might a student think so? Why isn't 100% the correct answer? What would this mean in the context of sunglasses? How would you respond to the student?

13. Prove the existence and uniqueness of $\sqrt[q]{a}$ when q is odd and $a \in \mathbb{R}, a \neq 0$.

Group homework

14. In your groups, type up a copy of a completed table for the handout *Summary: Defining Powers*.