2 Lesson 2

Opening Inquiry: Decisions have consequences

0. (a) What are the three power properties?

   (b) What is the definition of $a^{1/q}$?

   (c) What is the definition of $a^{p/q}$? (Look these up from your notes if you can't remember.)

1. What is $(-8)^{1/3}$?

2. Find three ways to simplify the expression $(-8)^{2/6}$.

Positive integer powers: Are products and powers well-defined?

1. How is a^x defined when $x \in \mathbb{N}_{>0}$?

2. Let the exponents x_1, x_2 be positive integers.
   Are products and powers of powers well-defined for any value of the base $a \in \mathbb{R}$?
   Complete the table. Mark ✓ for "Yes" and × for "No".

	Are these equalities always well-defined and true?	
When $x_1, x_2 \in \mathbb{N}_{>0}$ and ↓	$a^{x_1} a^{x_2} = a^{x_2} a^{x_1}$	$(a^{x_1})^{x_2} = (a^{x_2})^{x_1}$
$a > 0$		
$a = 0$		
$a < 0$		

3. Reasoning:

Adding and multiplying integers $\longleftrightarrow$ Products and powers of integer powers

1. How is a^x defined when $x = 0$ and $x \in \mathbb{Z}, x < 0$?
   When $x = 0$:

   When $x < 0$:

2. Here are some sample solutions to homework problems. What questions and comments do you have?

   - *Explain why* $-5 + 3 = 3 + -5$:
     - Think of -5 as five movements by -1, and $+3$ by three movements of $+1$.
     - The LHS (left hand side expression) is like starting at 0, moving -5, then moving $+3$.
       The RHS (right hand side expression) is like starting at 0, moving $+3$, then moving -5.
     - Either way, is the net result of moving -1 five times and $+1$ three times.
     - Movements of -1 paired with movements of 1 result in a net movement of 0, because -1 and $+1$ are additive inverses.
     - After pairing up the movements, there are two $+1$ movements remaining.
     - So $-5 + 3 = 3 + -5 = +2$.
   - *Explain why* $(-3) \times (-2) = 6$:
     - Multiplying -3 by -2 means taking the movement -3 twice and then switching the direction of the result.
     - Twice the movement -3 is the movement -6.
     - Switching the direction of the movement -6 gives us the movement $+6$.
     - So $(-3) \times (-2) = 6$.

3. Suppose your students have just learned that a^{-1} means $\frac{1}{a}$ and they are still struggling with simplifying fractional expressions with variables, as well as remembering what a^{-1} means.

   With the above in mind, work on (a)(b)(c).

   In all your explanations, be sure to:

   - Use the definition of a^{-1}
   - Use the definition of a^{-n}, where $n \in \mathbb{N}$
   - Use a structure as similar as possible to one of the explanations above.

   (a) What is an explanation that could help your students understand why $a^{-5}a^3 = a^3a^{-5}$?

   (b) What are $a^n a^{-m}$ and $a^{-m}a^n$ when $n > m$? When $n = m$? When $n < m$?

   (c) What is an explanation that could help your students understand why $(a^{-3})^{-2} = a^6$?

Natural and integer powers: Do power properties hold?

Let $x_1, x_2 \in \mathbb{Z}$.

Are the power properties well-defined for any value of the base $a \in \mathbb{R}$?

Complete the table. Mark ✓ for "Yes" and × for "No".

	(a) Are these equalities always well-defined and true? (b) What number line movement and location ideas do these most resemble?	
When $x_1, x_2 \in \mathbb{Z}$ and ↓	$a^{x_1} a^{x_2} = a^{x_2} a^{x_1}$	$(a^{x_1})^{x_2} = (a^{x_2})^{x_1}$
$a > 0$		
$a = 0$		
$a < 0$		

Reasoning:

DIVISIONS AND FRACTIONS $\longleftrightarrow$ RATIONAL POWERS

1. How is $a^{\frac{p}{q}}$ defined, for $p, q \in \mathbb{Z}, q \neq 0$?

   $p > 0$: $p = 0$: $p < 0$:

2. Here are some sample solutions to homework problems. What questions and comments do you have?

   - *There is only one number that can be the result of* $5 \div 3$. (*)
     - Let $x = 5 \div 3$. By definition of division, x is the quantity such that $x \times 3 = 5$.
     - Let $y \in \mathbb{R}$. Either $y = x$, $y > x$, or $y < x$.
     - Suppose $y < x$. Then $3y < 3x = 5$, so $y \neq 5 \div 3$.
     - Suppose $y > x$ Then $3y > 3x = 5$, so $y \neq 5 \div 3$.
     - Therefore $y = x$ and there is only one value such that $x = 5 \div 3$.
   - $5 \div 3$ *is the same number as* $\frac{1}{3} \cdot 5$.
     - Start with $x' = \frac{1}{3} \cdot 5$.
     - See what $3x'$ equals. If $3x' = 5$, then by (*), we can conclude $x' = x$.

$$\begin{aligned} 3x' = \frac{1}{3} \cdot 5 \cdot 3 &= \frac{1}{3} \cdot 3 \cdot 5 \quad \text{commutativity of multiplication in } \mathbb{Z} \\ &= 1 \cdot 5 \quad \text{definition of fraction} \\ &= 5 \quad \text{multiplicative identity} \end{aligned}$$

     - Hence $5 \div 3 = \frac{1}{3} \cdot 5$.

3. Suppose your students have just learned q-th roots, and they are still struggling with simplifying radical expressions and remembering what q-th roots are. They also need to review the definition of a^{-1} as $1/a$.

   With the above in mind, work on (a)(b)(c).

   In all your explanations, be sure to:

   - Use the definition of $a^{1/q}$
   - Use the definition of $a^{p/q} = (a^{1/q})^p$
   - Use the definitions of a^{-1} and a^{-n}, for $n \in \mathbb{N}$
   - Use a structure as similar as possible to the explanations above.

   (a) Help your students understand why $(7^5)^{1/3} = (7^{1/3})^5$.

   (b) Help your students understand why, in general, $(a^p)^{1/q} = (a^{1/q})^p$. (Assume a, p, q are all positive and $p, q \in \mathbb{N}$.)

   (c) Help your students understand why $(7^{-5})^{1/3} = (7^{1/3})^{-5}$.

RATIONAL POWERS OF NEGATIVE BASES

1. Analyze this claim: $(-1)^{\frac{1}{3}}(-1)^{\frac{1}{9}} = (-1)^{\frac{1}{3}+\frac{1}{9}}$. Is the equation a true statement? A false statement?

2. Now analyze this claim: $((-7)^5)^{1/3} = ((-7)^{1/3})^5$.

3. Using only the numbers $\pm1, \pm2, \pm4, \pm8$, come up with new examples that break the power of powers property.

4. Based on the definition of $a^{1/q}$ and $a^{p/q}$, does $((-7)^5)^{1/3} = ((-7)^{1/3})^5$? $((-7)^4)^{1/3} = ((-7)^{1/3})^4$? $((-7)^3)^{1/4} = ((-7)^{1/4})^3$? $((-7)^{10})^{1/4} = ((-7)^{1/4})^1 0$? For what p and q do these definitions seem to work for negative bases without problems?

RATIONAL POWERS: DO POWER PROPERTIES HOLD?

Let the exponents x_1, x_2 be rational.

Are the power properties well-defined for any value of the base $a \in \mathbb{R}$?

Complete the table. Mark ✓ for "Yes" and × for "No".

	(a) Are these equalities always well-defined and true? (b) What fraction ideas do these most resemble?	
When x_1, x_2 are zero or negative and ↓	$a^{x_1}a^{x_2} = a^{x_2}a^{x_1}$	$(a^{x_1})^{x_2} = (a^{x_2})^{x_1}$
$a > 0$		
$a = 0$		
$a < 0$		

Reasoning:

LIMITS IN HIGH SCHOOL CURRICULA

For homework, you investigated:

> Here are some places where limits and convergence can appear in high school curricula:
>
> - Geometry: *Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri?s principle, and informal limit arguments.* (Common Core G-GMD.1)
> - Pre-calculus: *Limits of functions and sequences.* (Lincoln Public School standards and course descriptions, https://home.lps.org/math/secondary/#)
> - AP Calculus: *The derivative of a function is defined as the limit of a difference quotient and can be determined using a variety of strategies. (College Board Course and Exam Description: AP Calculus AB and AP Calculus BC, https://apcentral.collegeboard.org/pdf/ap-calculus-ab-and-bc-course-and-exam-description.pdf)*
>
> (a) Describe an example of a convergent sequence that you might use while teaching one of geometry, pre-calc, or calculus.
> (b) What is an (informal) explanation of what it means for a sequence to converge or limit to something? This explanation should work across all these courses.

Here is a possible response. What questions and comments do you have?

(a)

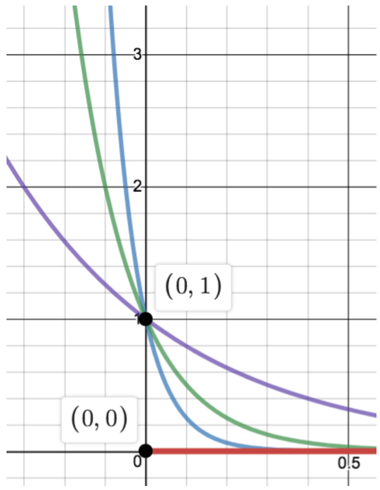
- *Geometry.* Given a circle with radius r. Let c_1 be the area of an equilateral triangle inscribed in the circle. Let c_2 be the area of a square inscribed in the circle. Let c_3 be the area of a regular pentagon inscribed in the circle. Keep on increasing the number of sides of the inscribed polygon in the circle, so c_n is the area of a regular $n+2$-gon inscribed in the circle. The limit of the sequence $\{c_1, c_2, \dots\}$ is the area of the circle. That is, $\lim_{n\to\infty} c_n = \pi r^2$.
- *Pre-calculus.* Let $c_1 = \frac{1}{2}, c_2 = \frac{3}{4}, c_3 = \frac{7}{8} \dots, c_n = 1 - \frac{1}{2^n}, \dots$. Then the sequence $\{c_1, c_2, \dots\}$ converges to 1. That is, $\lim_{n\to\infty} c_n = 1$.
- *AP Calculus.* The instantaneous rate of change of a function f at a point x_0 is $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$. Let $c_1 = \frac{f(x_0+0.1)-f(x_0)}{0.1}, c_2 = \frac{f(x_0+0.01)-f(x_0)}{0.01}, \dots c_n = \frac{f(x_0+10^{-n})-f(x_0)}{10^{-n}}, \dots$. When this sequence converges, we have $f'(x_0) = \lim_{n\to\infty} c_n$.

(b) *A sequence $\{c_0, c_1, c_2, \dots\}$ converges to a real number if, as n gets larger, c_n and c_{n-1} get closer and closer to each other to the point where there is practically 0 difference between them.*

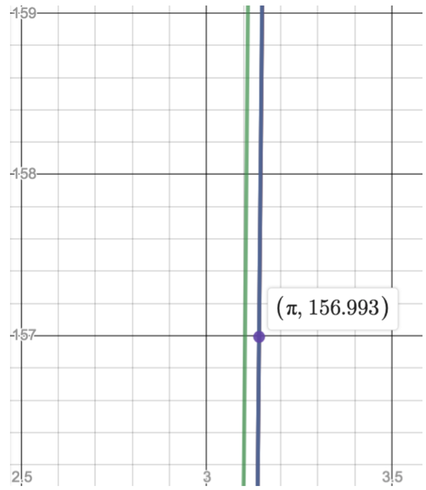
LIMITS OF POWERS

For homework, you graphed variations of 0^x and 5^x.

The graphs you sketched looked like:



$0^x, (0.1)^x, (0.001)^x (0.000001)^x$
near $x = 0$



$5^x, (5.1)^x, (5.001)^x (5.000001)^x$
near $x = \pi$

(a) Look at the first set of variations. As we move from the graph of 0.1^x to 0.001^x to 0.000001^x, what happens to the instantaneous slope at the point $(0, 1)$?

(b) What do you think the slope of $\lim_{n\to\infty} \left(\frac{1}{10^n}\right)^x$ would be, near $x = 0$?

(c) If the function 0^x could be assigned a slope at $x = 0$, what should it be?

(d) How do you think the slope of $\lim_{n\to\infty} \left(5 + \frac{1}{10^n}\right)^x$ compares to the slope of 5^x, near $x = \pi$?

Takeaways:

Real powers: Do power properties hold?

1. How is a^x defined when $x \in \mathbb{R}$?

2. Let the exponents $x_1, x_2 \in \mathbb{R}$.
   Are the power properties well-defined for any value of the base $a \in \mathbb{R}$?
   Complete the table. Mark ✓ for "Yes" and $\times$ for "No".

When x_1, x_2 are zero or negative and ↓	Are these equalities always well-defined and true? $a^{x_1}a^{x_2} = a^{x_2}a^{x_1}$	$(a^{x_1})^{x_2} = (a^{x_2})^{x_1}$
$a > 0$		
$a = 0$		
$a < 0$		

3. Reasoning:

POWER PROPERTIES: FINAL DRAFT

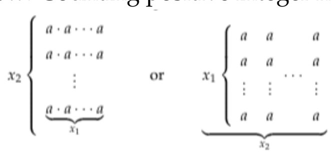
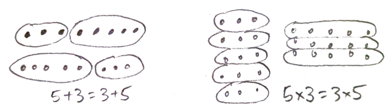
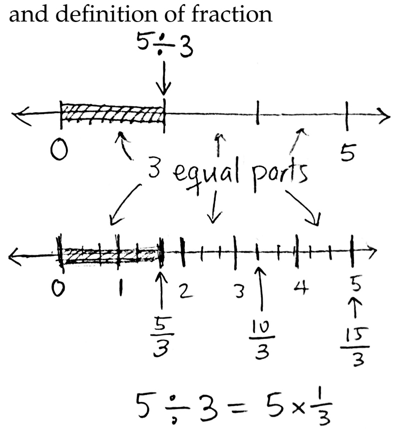
Property 2.1 (Power Properties)**.** Let $a \in \mathbb{R}, a > 0$ and $x, x_1, x_2 \in \mathbb{R}$. Then exponential expression of the form a^x satisfy the following properties:

- (1st power) $a^1 = a$
- (product of powers property) $a^{x_1+x_2} = a^{x_2+x_1} = a^{x_1}a^{x_2}$
- (power of a power property) $(a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1x_2}$

Note: Power properties allow us to say that, for positive bases, exponent addition and multiplication are commutative, associative, and satisfy the distributive property.

When $a \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{N}_{>0}$, the power properties hold. When $a \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{Z}$, the power properties still hold as long as $a \neq 0$. But when $x_1, x_2 \in \mathbb{Q}$ or $x_1, x_2 \in \mathbb{R}$, then we need to restrict the base to the domain $a > 0$.

EXPLAINING POWER PROPERTIES: PARALLELS IN K-12 MATHEMATICS

Explaining power properties	Parallel in K-12
Power properties for positive integer exponents How: Counting positive integer number of real factors 	Commutativity of addition and multiplication of positive integers How: Counting dots 
Power properties for integer exponents How: Multiplying and dividing factors; pair up multiplicative inverses.	Commutativity of addition and multiplication of integers How: Location/movement on number line; pair up additive inverses
Power properties for rational non-integer exponents How: Compare powers; use definition of q-th root, which is based on definition of fraction	Arithmetic properties of division and fractions (e.g., $p \div q = \frac{1}{q} \cdot p$) How: Comparing lengths; use definition of division and definition of fraction 
Power properties for real non-rational exponents How: Use sequences of rational powers; use meaning of convergent sequence for values and rates of change	Limits of sequences, in pre-calculus and calculus How: Meaning of convergence