Part I

Correspondence View on Exponentiation

1 Lesson 1

TAKEAWAYS: DEFINING NUMBERS

What are your thoughts and questions about the number line activity we just did?

Two common ways to think of numbers are:

-
-

How did we just use both these ways?

Let's look at one of these ways: (next side)

movement	interpretation & why it makes sense
1	(after placing the *locations* "0", "1") *Starting assumption*: Moving 1 means moving to the right by the distance between the locations 0 and 1.
positive integers $n \in \mathbb{N}$	
-1	
negative integers $-n$, where $n \in \mathbb{N}$	
unit fractions $1/n$, where $n \in \mathbb{Z}_{\neq 0}$	positive n: negative n:
rational numbers p/n, where $p, n \in \mathbb{Z}$, $n \neq 0$	
(tba)	

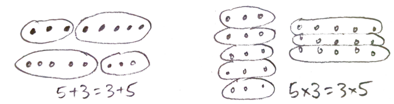
How would you describe a relationship between the two ways?

REFERENCE: HIGHLIGHTS OF NUMBER AND OPERATION, K-8

Natural numbers
$0, 1, 2, 3, \ldots$

n is defined as repeated addition of n 1's

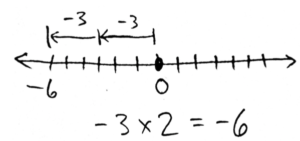
$$n = \underbrace{1 + 1 + \cdots + 1}_{n}.$$



Arithmetic properties can be justified by counting dots.

Negative natural numbers
$-1, -2, -3, \ldots$

$-n$ is defined as the additive inverse of n. This means it is the number such that $-n + n = 0$.
Movement by $-n$ is defined as moving the same distance as n, but in the opposite direction.
Multiplication by $-n$ is defined as taking n times something, and then switching the direction of the product.
Arithmetic properties can be justified using location and movement on the number line.



Rational numbers
p/n, where
$p, n \in \mathbb{N}, n \neq 0$

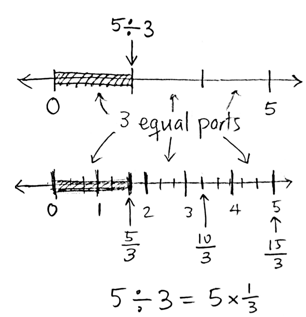
Rational numbers are initially defined as fractions:

- Identify the whole
- Divide whole into n equal parts.
- $1/n$ is one of the equal parts.
- n of those parts is the whole.
- p of those equal parts is p/n.

Later, they are defined using division:

- p/n is the result of dividing a quantity p into n equal parts.

Later, $-p/n$ is defined as the additive inverse of p/n. This means it is the number such that $-\frac{p}{n} + \frac{p}{n} = 0$.
Finally, $1/n$ is also described as the multiplicative inverse of n, meaning it is the number such that $\frac{1}{n} \cdot n = 1$.
Arithmetic properties can be justified using the number line.



Real numbers

Real numbers are composed of rational numbers and irrational numbers. Irrational numbers are all the numbers on the number line that are not rational.

JUSTIFYING ARITHMETIC RESULTS USING NUMBER LINE MODELS

1. (a) The diagram in the *Highlights* table shows $-3 \times 2 = -6$, where the first number is interpreted as a movement to be multiplied. What questions or comments do you have about this explanation or diagram?

   (b) Using the models for movement and multiplication described in *Highlights*, explain why $2 \times (-3) = -6$. Draw some diagrams to help explain.

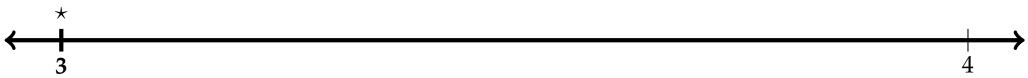
2. (a) Explain how to use the definition of division to estimate where $5 \div 3$ is.

   (b) Explain how to use the definition of fraction to estimate where $1/3$ is, and then where $5/3$ is.

   (c) What are some reasons that elementary students might struggle with the idea that $5/3$ is equal to $5 \div 3$?

Where is π located, exactly?

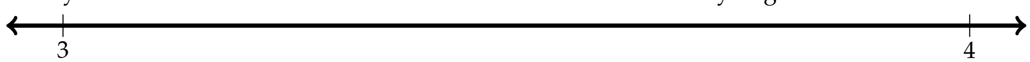
Suppose you ($\star$) are a point on the number line, and you are currently located at 3.



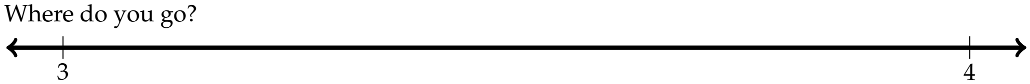
For each of the following, determine whether you need to move or not. If you don't need to move, draw where you currently are. If you do need to move, draw where you move to, and label the location.

0. You want to be within 1 of the *exact* location of π. Where do you go?

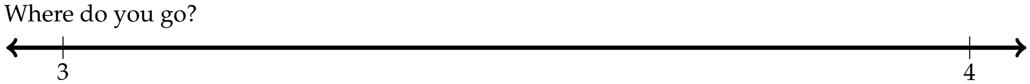
1. Now you want to be within 0.1 of the *exact* location of π. Where do you go?



2. Now you want to be within $0.01 = 1/100$ of the *exact* location of π.
   Where do you go?



3. Now you want to be within $0.001 = 1/1000$ of the *exact* location of π.
   Where do you go?



Here is an approximation of π:

$\pi \approx$ 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535940812848 1 . . .

10. Now you want to be within $0.\underline{000}\ \underline{000}\ \underline{000}\,1 = 1/10^{10}$ of the *exact* location of π.
Where do you go?

n. Now you want to be within $0.\underbrace{00\ldots0}_{n \text{ zeros}}1 = 1/10^{n}$ of the *exact* location of π.
Where do you go?

Here is an approximation of π:

$\pi \approx$ 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535940812848 1 …

Takeaways: Where is π, exactly?

What are some takeaways for you from this experience? What questions do you have?

Stacking Sunglasses

Based on our work so far, we can say that:

a person wearing ________ sunglasses each with 5% tint	can see ________ of the color around them (% visible light transmission)
single	$0.95 = 95\%$
quintuple	$(0.95)^5 = 77.4\%$
hextuple	$(0.95)^6 = 73.5\%$
septuple	$(0.95)^7 = 69.8\%$

Using the expressions in the table, find <u>three different ways</u> to compute the answer to this question:

> A person wears 18 sunglasses, each at 5% tint.
> How much of the color around them can they see?

SUMMARY: DEFINING POWERS

Exponential expression. In expressions such as $(0.95)^n$, we refer to 0.95 as the **base** and n as the **exponent**. The expression $(0.95)^n$ is referred to as a **power**.

Power Properties (draft). Exponential expressions satisfy the following properties:

- (1st power) $a^1 = a$
- (product of powers property) $a^{x_1+x_2} = a^{x_1}a^{x_2}$
- (power of a power property) $(a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1 x_2}$

Working definitions of powers:

Exponent type	Working definition of power	Why it makes sense
1	Define a^1 as a.	Starting assumption: First power property
positive integer $x \in \mathbb{N}_{>0}$	Define a^x as the product of multiplying x a's: $a^x = \underbrace{a \cdot a \cdots a}_{x}$. *Examples:* $(0.95)^2 = 0.95 \times 0.95 = 0.9025$ $0^3 = 0 \cdot 0 \cdot 0 = 0$. Note: $a^x = 0 \iff a = 0$.	Starting assumption: Product of powers property. $\underbrace{a^1 \cdot a^1 \cdots a^1}_{x} = a^{1+1+\cdots+1} = a^x$
(tba)		

For the exponent types below, assume $a \neq 0$.

-1	Define a^{-1} as $1/a$, the multiplicative inverse of a.	Homework
negative integer $-x$, where $x \in \mathbb{N}_{>0}$	Define a^{-x} as $(a^{-1})^x$. We define this as equivalent to $\left(\frac{1}{a}\right)^x$ and $\frac{1}{a^x}$.	Homework

For the exponent types below, assume $a \neq 0$.

Exponent type	Working definition of power	Why it makes sense

DEFINING a^0

What is a good definition for a^0? How would you defend your definition?

Definition for a^0 (draft):

Reasoning for definition:

MORE ON ZEROS

1. (a) Does this sequence converge? If so, to what? If not, why not?

$$1^0, \quad (\frac{1}{2})^0, \quad (\frac{1}{3})^0, \quad (\frac{1}{4})^0, \ldots$$

   (b) Based on what you have found, what should 0^0 equal?

2. (a) Does this sequence converge? If so, to what? If not, why not?

$$0^1, \quad 0^{\frac{1}{2}}, \quad 0^{\frac{1}{3}}, \quad 0^{\frac{1}{4}}, \ldots$$

   (b) Based on what you have found, what should 0^0 equal?

Fractional powers opening example

What is $8^{5/3}$? Find at least two ways to explain why.

Defining $a^{1/q}$

(a) Write a draft definition for $a^{1/q}$.

(b) Based on your definition, what should $81^{\frac{1}{4}}$ be?

(c) Based on your definition, what would $(-81)^{\frac{1}{2}}$ be?

Relationship between q-th roots and $a^{1/q}$

There are two related definitions, which we will fill in as a class:

Definition 1:

Definition 2:

Why these definitions are both necessary:

Why the second definition is well-defined:

Why the definition of $a^{1/q}$ makes sense using power properties:

DEFINING $a^{p/q}$

Write a draft definition for $a^{p/q}$:

Use exponential properties to justify your definition:

Reflection on Defining Numbers and Powers

Read through our work on *Takeaways: Defining Numbers* and *Summary: Defining Powers*.

1. What parallels do you see when you compare ...
   (a) ... the definitions of negative integer movements and negative integer powers?
   (b) ... the definitions of rational number movements and rational powers?

2. Looking across the table, what differences do you see?

RAISING TO THE π-TH POWER

1. To begin, let's take stock of what we have been thinking about.
   How would you explain to a middle school student:

   - What does 5^3 mean?

   - What does $5^{3\frac{1}{10}}$ mean?

   - What does $5^{3\frac{14}{100}}$ mean?

2. Building on these ideas, how would you explain to a middle school student:

   - What does 5^π mean? How would you find it?

Sequence of Powers

What do you notice? What do you wonder?

Δx	x	5^x
	3	625
$0.1 \leq 1$	3.1	146.8273679
$0.04 \leq \frac{1}{10}$	3.14	156.5906452
$0.002 \leq \frac{1}{10^2}$	3.141	157.0955032
$0.0004 \leq \frac{1}{10^3}$	3.1416	156.9944015
$0.00001 \leq \frac{1}{10^4}$	3.14159	156.9918748