Mathematics Of Doing, Understanding, Learning, and Educating Secondary Schools

MODULE(S^2): Algebra for Secondary Mathematics Teaching

Module 2: Exponentiation, Exponential Growth, and Complex Arithmetic

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Part I

Correspondence View on Exponentiation

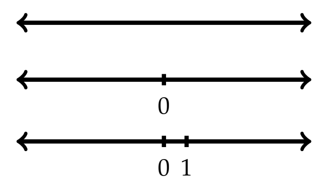
1 Constructing Powers from Properties of Exponents (Lesson 1) (Length: ~2.5 hours)

Defining Numbers

OPENING INQUIRY

Defining numbers

We are going to make a number line.



To begin, let's choose a place for the origin, 0.

Now let's choose a unit distance.

I'm going to move to zero.

Now I'd like you to give me instructions for how to move to "5".

Tell me how to move "-3" from where I am.

Now give me instructions how to tell whether I have moved $1/3$ from where I am.

Now tell me how to move $7/3$ from where I am.

Now, in partners, talk about how to tell me how to move $-5/3$ from where I am.

Now I want to make sure I move exactly 0.1. How do I do that? Talk in partners.

Finally, talk in partners about how to move exactly 0.04. Use the thinking about fractions that we have already developed.

DEFINING INTEGERS AND RATIONALS ON THE NUMBER LINE

Let's reflect on what we just did, using number and operation ideas that our students learned in K-8.

Takeaways: Defining numbers

What are your thoughts and questions about the number line activity we just did

Two common ways to think of numbers are:

-
-

How did we just use both these ways?

Let's look at one of these ways:

movement	interpretation & why it makes sense
1	(after placing the *locations* "0", "1") *Starting assumption*: Moving 1 means moving to the right by the distance between the locations 0 and 1.
positive integers $n \in \mathbb{N}$	
-1	
negative integers $-n$, where $n \in \mathbb{N}$	
unit fractions $1/n$, where $n \in \mathbb{Z}_{\neq 0}$	positive n: negative n:
rational numbers p/n, where $p, n \in \mathbb{Z}$, $n \neq 0$	
(tba)	

How would you describe a relationship between the two ways?

Takeaways. Two common ways to think of numbers are:

- Location (on the number line)
- Movement (on the number line)

We used both of these ideas just now:

- We used location when we said we were starting at 0.
- You used movement to figure out how to get all integer and rational distances.

In particular:

- To move $\underline{1}$: After placing the *location* "1", we start with the assumption that *movement* $\underline{1}$ means moving to the right, by the distance between location 0 and location 1.

  We use the locations of 0 and 1, and movement by 1, to build up everything else.

Then, to move:

- $\underline{\text{positive integer } n}$, we repeated the movement 1, by n times. (Ex. 5)

  Why it makes sense: Because n is repeated addition of n $1's$: $n = \underbrace{1 + 1 + \cdots + 1}_{n}$.

- $\underline{-1}$ we moved by 1, but to the *left*.

  Why it makes sense: -1 is defined as the additive inverse of 1, meaning $-1 + 1 = 0$. So moving -1 and then moving 1, or vice versa, should result in zero net movement. So moving -1 should be moving the same distance as 1, but in the opposite direction.

- $\underline{\text{negative integer } -n}$, we repeated the movement -1, by n times. (Ex. -3).

  Why it makes sense: $-n$ is defined as the additive inverse of n, meaning $-n + n = 0$. So moving $-n$ and then moving n, or vice versa, should result in zero net movement. So moving $-n$ should be moving the same distance as n, but in the opposite direction.

- $\underline{\text{positive unit fractions } \frac{1}{n}}$, we use the movement that repeated n times gives 1. ("Unit fraction" means $n \in \mathbb{Z}$.) (Ex. $1/3, 1/10, 1/25, 1/100$.)

  Why it makes sense: $1/n$ is defined as the quantity you get when you divide 1 into n equal parts. So moving $1/n$, n times, should give you the same movement as 1.

- <u>negative unit fraction $-\frac{1}{n}$</u>, we move by $1/n$, but to the *left*. (Ex. $-1/3$, as part of moving $-5/3$.)

  Why it makes sense: $-\frac{1}{n}$ is defined as the additive inverse of $\frac{1}{n}$, so we should move the same distance as $1/n$, but in the opposite direction.

- <u>p/n</u>, we move by $1/n$, by p times when $p > 0$ and p times to the opposite direction when $p < 0$. (Ex. $4/100$, $-5/3$.)

  Why it makes sense: p/n is defined as the quantity you get when you take p of $1/n$.

The reasoning we used is consistent with how numbers are constructed and properties are justified in elementary and middle school.

One of our goals this lesson is to learn these explanations, and you will have some of these explanations for homework. This is because they will become templates for giving explanations about exponentiation, which we will address later this lesson.

Here is a reference for these kinds of properties.

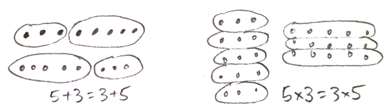
Reference: Highlights of Number and Operation, K-8

Natural numbers
$0, 1, 2, 3, \ldots$

n is defined as repeated addition of n 1's

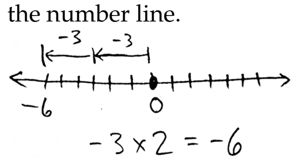
$$n = \underbrace{1 + 1 + \cdots + 1}_{n}.$$

Arithmetic properties can be justified by counting dots.



Negative natural numbers
$-1, -2, -3, \ldots$

$-n$ is defined as the additive inverse of n. This means it is the number such that $-n + n = 0$.
Movement by $-n$ is defined as moving the same distance as n, but in the opposite direction.
Multiplication by $-n$ is defined as taking n times something, and then switching the direction of the product.
Arithmetic properties can be justified using location and movement on the number line.



Rational numbers
p/n, where $p, n \in \mathbb{N}, n \neq 0$

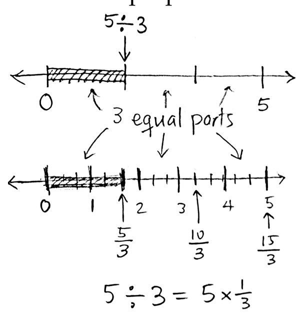
Rational numbers are initially defined as fractions:

- Identify the whole
- Divide whole into n equal parts.
- $1/n$ is one of the equal parts.
- n of those parts is the whole.
- p of those equal parts is p/n.

Later, they are defined using division:

- p/n is the result of dividing a quantity p into n equal parts.

Later, $-p/n$ is defined as the additive inverse of p/n. This means it is the number such that $-\frac{p}{n} + \frac{p}{n} = 0$.
Finally, $1/n$ is also described as the multiplicative inverse of n, meaning it is the number such that $\frac{1}{n} \cdot n = 1$.
Arithmetic properties can be justified using the number line.



Real numbers

Real numbers are composed of rational numbers and irrational numbers. Irrational numbers are all the numbers on the number line that are not rational.

Justifying arithmetic results using number line models

1. (a) The diagram in the *Highlights* table shows $-3 \times 2 = -6$, where the first number is interpreted as a movement to be multiplied. What questions or comments do you have about this explanation or diagram?

   (b) Using the models for movement and multiplication described in *Highlights*, explain why $2 \times (-3) = -6$. Draw some diagrams to help explain.

2. (a) Explain how to use the definition of division to estimate where $5 \div 3$ is.

   (b) Explain how to use the definition of fraction to estimate where $1/3$ is, and then where $5/3$ is.

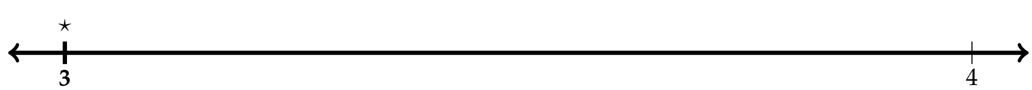
   (c) What are some reasons that elementary students might struggle with the idea that $5/3$ is equal to $5 \div 3$?

DEFINING IRRATIONAL NUMBERS ON THE NUMBER LINE

But there are more numbers than this. What's missing? Irrational numbers. To get an idea of how we can work with irrationals using the metaphors of location and movement, let's look at how to find π.

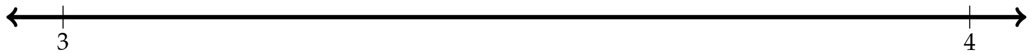
Where is π located, exactly?

Suppose you ($\star$) are a point on the number line, and you are currently located at 3.



For each of the following, determine whether you need to move or not. If you don't need to move, draw where you currently are. If you do need to move, draw where you move to, and label the location.

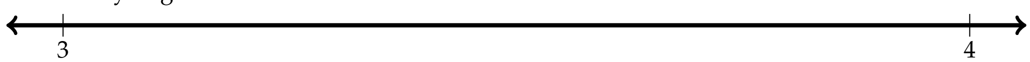
0. You want to be within 1 of the *exact* location of π. Where do you go?



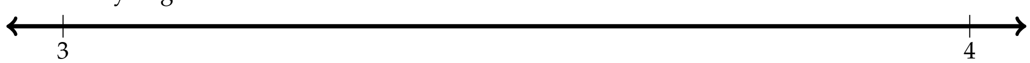
1. Now you want to be within 0.1 of the *exact* location of π. Where do you go?

2. Now you want to be within $0.01 = 1/100$ of the *exact* location of π.
Where do you go?

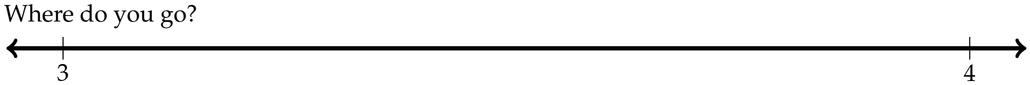
3. Now you want to be within $0.001 = 1/1000$ of the *exact* location of π.
Where do you go?



10. Now you want to be within $0.\underline{000}\,\underline{000}\,\underline{000}\,1 = 1/10^{10}$ of the *exact* location of π.
Where do you go?



n. Now you want to be within $0.\underbrace{00\ldots0}_{n \text{ zeros}}1 = 1/10^n$ of the *exact* location of π.
Where do you go?



Here is an approximation of π:

$\pi \approx$ 3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117067982148086513282306647093844609550582231725359408128481 ...

Takeaways: Where is π, exactly?

What are some takeaways for you from this experience? What questions do you have?

We have learned:

- To move by an irrational x, we construct a sequence of rational numbers $x_0, x_1, x_2, \ldots$ that get closer and closer to that irrational.

  Why it makes sense: In the limit, $\lim_{n\to\infty} x_n = x$, so movements by those rationals converge to movement by that irrational.

Relationship between location and movement:

- Use locations of 0, 1 to define movement by 1.
- Then movement by any quantity r from 0 gives the location r.

Now let's begin our work on exponentiation.

Power Properties: Assumptions for Defining Powers

Eliciting Properties of Powers

GQ

Double Sunglasses

Play this slide show: https://drive.google.com/drive/folders/1IoJw1lZX5hNuvnLsrzbVOTNk7ntqXOHF.

Act I. What questions come to mind?

Act II. What information do you need to know?

Act III. Which is closest to the correct answer?

100% 50% 75% 25%

What questions and comments do you have?

Sequel. How many 5% tint sunglasses would it take to reach 60% tint?

Stacking Sunglasses

Based on our work so far, we can say that:

a person wearing ________ sunglasses each with 5% tint	can see ________ of the color around them (% visible light transmission)
single	$0.95 = 95\%$
quintuple	$(0.95)^5 = 77.4\%$
hextuple	$(0.95)^6 = 73.5\%$
septuple	$(0.95)^7 = 69.8\%$

Using the expressions in the table, find <u>three different ways</u> to compute the answer to this question:

> A person wears 18 sunglasses, each at 5% tint.
> How much of the color around them can they see?

So how much is $(0.95)^{18}$? It is close to 0.397. This is the number of 5% tint sunglasses needed to be equivalent to at least 60% tint!

Power Properties: Draft

In the sunglasses task, we used exponential expressions:

Definition 1.1 (Exponential expression)**.** In expressions such as $(0.95)^n$, we refer to 0.95 as the **base** and n as the **exponent**. The expression $(0.95)^n$ is referred to as a **power**.

This use of "power" and "exponent" is used in K-12 texts.

This use distinguishes between components of expression and the whole expression.

It allows us to say key properties of exponentiation more clearly, for instance: "multiplying powers means adding exponents", "logarithms map powers to exponents", "exponentiation maps exponents to powers'."

Your reasoning on the *Stacking Sunglasses* task relied on the following properties:

Property 1.2 (Power Properties: Draft)**.** Exponential expressions satisfy the following properties:

- (1st power) $a^1 = a$
- (product of powers property) $a^{x_1+x_2} = a^{x_2+x_1} = a^{x_1}a^{x_2}$
- (power of a power property) $(a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1 x_2}$

Note: Power properties allow us to say that exponent addition and multiplication are commutative, associative, and satisfy the distributive property.

Example. $(0.95)^1 = 0.95$ (1st power)
$(0.95)^{18} = (0.95)^5(0.95)^6(0.95)^7$, $(0.95)^{18} = \underbrace{(0.95^1)(0.95^1)\ldots(0.95^1)}_{18}$ (product of powers)
$(0.95)^{18} = ((0.95)^6)^3$ (power of a power)

Defining Powers, Inspired by Defining Numbers

Summary: Defining Powers

Exponential expression. In expressions such as $(0.95)^n$, we refer to 0.95 as the **base** and n as the **exponent**.

The expression $(0.95)^n$ is referred to as a **power**.

Power Properties (draft). We start with the assumption that exponential expressions satisfy the following properties:

- (1st power) $a^1 = a$
- (product of powers property) $a^{x_1+x_2} = a^{x_1}a^{x_2}$
- (power of a power property) $(a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1 x_2}$

Working definitions of powers:

Exponent type	Working definition of power	Why it makes sense
1	Define a^1 as a.	First power property
positive integer $x \in \mathbb{N}_{>0}$	Define a^x as the product of multiplying x a's: $a^x = \underbrace{a \cdot a \cdots a}_{x}$. *Examples:* $(0.95)^2 = 0.95 \times 0.95 = 0.9025$ $0^3 = 0 \cdot 0 \cdot 0 = 0$. Note: $a^x = 0 \iff a = 0$.	Product of powers property: $\underbrace{a^1 \cdot a^1 \cdots a^1}_{x} = a^{1+1+\cdots+1} = a^x$

For the exponent types below, assume $a \neq 0$.

-1	Define a^{-1} as $1/a$, the multiplicative inverse of a.	Homework
negative integer $-x$, where $x \in \mathbb{N}_{>0}$	Define a^{-x} as $(a^{-1})^x$. We define this as equivalent to $\left(\frac{1}{a}\right)^x$ and $\frac{1}{a^x}$.	Homework
$\vdots$	$\vdots$	$\vdots$

Positive integer powers

What we did with the sunglasses task helped us with defining exponential expression for positive integer powers.

Negative integer powers

Next, we looked at negative integer powers. We'll give the definition in class, and you'll have the reasoning for homework.

.

Reasoning for definition. For homework. □

Now let's look at other powers.

Zero powers

Defining a^0

What is a good definition for a^0? How would you defend your definition?

Definition for a^0 (draft):

Reasoning for definition:

Definition 1.3 (Working definition for a^0 for $a \neq 0$.)**.** Let $a \in \mathbb{R}, a \neq 0$. We define a^0 as 1.

Reasoning for definition. We are trying to solve for a^0. (We'll put a box around it to remind us that it is the mystery value.) Let x be any positive integer.

$$\begin{aligned} a^x \boxed{a^0} &= a^{x+0} \quad \text{(product of power property)} \\ &= a^x \quad (x+0=0, \text{by additive identity of } \mathbb{R}) \\ \boxed{a^0} &= \frac{a^x}{a^x} \quad \text{(division in } \mathbb{R}\text{, assuming } a^x \neq 0) \\ \boxed{a^0} &= 1 \quad \text{(multiplicative inverse in } \mathbb{R}\text{, assuming } a^x \neq 0) \end{aligned}$$

So we define $a^0 = 1$, because that's what you get when you use the product of power property. □

More on zeros

1. (a) Does this sequence converge? If so, to what? If not, why not? $1^0, \quad (\frac{1}{2})^0, \quad (\frac{1}{3})^0, \quad (\frac{1}{4})^0, \ldots$
   (b) Based on what you have found, what should 0^0 equal?
2. (a) Does this sequence converge? If so, to what? If not, why not? $0^1, \quad 0^{\frac{1}{2}}, \quad 0^{\frac{1}{3}}, \quad 0^{\frac{1}{4}}, \ldots$

(b) Based on what you have found, what should 0^0 equal?

Takeaways:

- There's something weird going on when 0 is both the base and the exponent. We can come back to this, but for now, let's stick with the case where the base is positive.
- One hint that we were going to run into trouble is in using the derivation $a^0 = a^x/a^x = 1$, where $x \in \mathbb{N}_{>0}$. In doing this, we are already saying that a can't be zero.

UNIT FRACTION POWERS

Fractional powers opening example

What is $8^{5/3}$? Find at least two ways to explain why.

Using the power of product rule, two possible strategies for this are:

$$8^{5/3} = (8^{1/3})^5$$
$$8^{1/3} = 2$$
$$\text{So } (8^{1/3})^5 = 2^5 = 32.$$

$$8^{5/3} = (8^5)^{1/3}$$
$$8^5 = 2^{15}$$
$$\text{So } (8^5)^{1/3} = (2^{15})^{1/3} = 2^5 = 32.$$

In both cases, we use the idea that $8^{1/3}$ should be 2, because

$$(8^{1/3}) \cdot (8^{1/3}) \cdot (8^{1/3}) = 8^1 = 8$$

So finding $8^{1/3}$ means finding $a \in \mathbb{R}$ such that $a^3 = 8$.

Defining $a^{1/q}$

(a) Write a draft definition for $a^{1/q}$.

(b) Based on your definition, what should $81^{\frac{1}{4}}$ be?

(c) Based on your definition, what would $(-81)^{\frac{1}{2}}$ be?

Some things to consider:

- Numerical expressions should only mean one number.
- If there is more than one reasonable choice of number, we need a way to decide which choice to pick as our definition.
- For this reason, whenever q is even, we need to decide whether $a^{1/q}$ means the positive or negative root, assuming it exists.

Relationship between q-th roots and $a^{1/q}$

There are two related definitions, which we will fill in as a class:

- **Definition 1:**
- **Definition 2:**

Why these definitions are both necessary:

Why the second definition is well-defined:

Why the definition of $a^{1/q}$ makes sense using power properties:

Definition 1.4 (q-th root)**.** Given $a \in \mathbb{R}$, and $q \in \mathbb{N}, q \neq 0$, we say that **$r$ is a q-th root of a** if $r^q = a$.

We give the following definition, and then discuss why the definition specifies only one number.

Definition 1.5 (Working definition for $a^{1/q}$)**.** Given $a \in \mathbb{R}$, and $q \in \mathbb{N}, q \neq 0$, we say that $\underline{a^{1/q}}$ is the positive q-th root of a, if one exists. Otherwise, if there is no positive q-th root but there is a negative q-th root, we say it is the negative q-th root of a.

Note: All positive real numbers have a unique positive q-th root, as shown in Theorem 1.6. Using similar reasoning, we can show that all negative numbers have a unique negative q-th root when q is odd. So we define the following to mean the same thing:

- "the positive real q-th root of a, when one exists; otherwise the negative real q-th root of a, if one exists and there is no positive real q-th root"
- $\sqrt[q]{a}$
- $a^{1/q}$

Theorem 1.6 (existence and uniqueness of positive q-th root for real numbers)**.** *For all positive real x and positive integer q, there exists a unique positive q-th root of x.*

For all negative real x, there exists a unique negative q-th root of x when q is odd.

Partial proof, Only for first part of theorem. If q is positive, and x is a positive real number, there is a unique positive q-th root of x. To show existence, we can use the Intermediate Value Theorem on the function $f(x) = x^q$. To show uniqueness of positive roots, we can use proof by contradiction. Suppose that $y > 0$ and $z > 0$ are both q-th roots of x, with $y \neq z$. Then $y^q = z^q = x$. But if $y \neq z$, then $y > z$ or $y < z$. Without loss of generality, let $y > z$. Then $y^q > z^q$. This contradicts trichotomy of real numbers, as it cannot be true that both $y^q = z^q$ and $y^q > z^q$. So there is a unique positive q-th root. If q is negative, and x is a positive real number, then $1/x$ is a positive real number. By consequence of the above argument, $1/x$ has a unique positive q-th root. ☐

Why Definition 1.5, for unit fraction powers makes sense, using power properties:

Reasoning for working definition for $a^{1/q}$. We are trying to solve for $a^{1/q}$.

$$\begin{aligned}
\underbrace{\boxed{a^{1/q}} \cdot \boxed{a^{1/q}} \cdots \boxed{a^{1/q}}}_{q} &= a^{\overbrace{\frac{1}{q}+\cdots+\frac{1}{q}}^{q}} \quad \text{(Product of Powers)} \\
&= a^1 \quad \text{(Definition of Fraction)} \\
(\boxed{a^{1/q}})^q &= 1 \quad \text{(Definition of Positive Integer Power; First Power)}
\end{aligned}$$

Hence $a^{1/q}$ satisfies the definition of q-th root. We want it to only mean one thing, so we pick the positive q-th root when it exists, and if that doesn't exist, we pick the negative q-th root. ☐

Rational powers

Defining $a^{p/q}$

Write a draft definition for $a^{p/q}$:

Use exponential properties to justify your definition:

Let's record a summary of what we have figured out.

Exponent type	Working definition of power & Why it makes sense
1, positive integer, -1, negative integer	$a^1 = a$ $a^x = \underbrace{a \cdot a \cdots a}_{x}$, $a^{-1} = 1/a$, where $1/a$ is the multiplicative inverse of a $a^{-x} = (a^{-1})^x$ or $(\frac{1}{a})^x$ or $\frac{1}{a^x}$ Why: Homework
0	$\left.\begin{array}{lll} a \neq 0 & \implies & a^0 = 1 \\ a = 0 & \implies & a^0 \text{ is indeterminate.} \end{array}\right\}$ Why: See Zero Powers
	Below, assume $a \neq 0$
unit fraction $1/q$ $q \in \mathbb{Z}_{\neq 0}$	$\left.\begin{array}{lll} q > 0 & \implies & a^{1/q} = \text{positive } q\text{-th root of } a\text{, if it exists} \\ q < 0 & \implies & a^{1/q} = \text{positive } q\text{-th root of } 1/a\text{, it if exists} \end{array}\right\}$ Why: Homework
rational number p/q $p, q \in \mathbb{Z}, q \neq 0$	$a^{p/q} = (a^{1/q})^p$ Why: If $p > 0$, then $a^{p/q} = \underbrace{(a^{1/q}) \cdot (a^{1/q}) \cdots (a^{1/q})}_{p}$. (power of powers, definition of positive integer power) If $p = 0$, then $a^{p/q} = 1$. (1st power) If $p < 0$, then $a^{p/q} = \frac{1}{\underbrace{(a^{1/q}) \cdot (a^{1/q}) \cdots (a^{1/q})}_{\lvert p \rvert}}$. Homework

Reflection on Defining Numbers and Powers

Read through our work on *Takeaways: Defining Numbers* and *Summary: Defining Powers*.

1. What parallels do you see when you compare ...
   (a) ... the definitions of negative integer movements and negative integer powers?
   (b) ... the definitions of rational number movements and rational powers?
2. Looking across the table, what differences do you see?

Takeaways:

- Parallels can be found in how movements and powers are defined.
  - We define negative integer movement by (additively) inverting positive integer movements. We define negative integer powers by (multiplicatively) inverting positive integer powers.
  - We define rational number movement by (additively) partitioning integer movements. We define rational powers by (multiplicatively) partitioning integer powers.
- **A key difference** is that when defining numbers, we use addition, and **when we define powers, we use multiplication**.
- Exponential reasoning is often referred to as **multiplicative reasoning**. One of the struggles that students encounter is the shift from additive reasoning to multiplicative reasoning.

IRRATIONAL POWERS

There is one more kind of power we need to figure out: powers of irrationals.

Raising to the π-th power

1. To begin, let's take stock of what we have been thinking about.
   How would you explain to a middle school student:
   - What does 5^3 mean?
   - What does $5^{3\frac{1}{10}}$ mean?
   - What does $5^{3\frac{14}{100}}$ mean?
2. Building on these ideas, how would you explain to a middle school student:
   - What does 5^π mean? How would you find it?

Sequence of Powers

What do you notice? What do you wonder?

Δx	x	5^x
	3	625
$0.1 \leq 1$	3.1	146.8273679
$0.04 \leq \frac{1}{10}$	3.14	156.5906452
$0.002 \leq \frac{1}{10^2}$	3.141	157.0955032
$0.0004 \leq \frac{1}{10^3}$	3.1416	156.9944015
$0.00001 \leq \frac{1}{10^4}$	3.14159	156.9918748

What this example illustrates:

- $5^\pi \approx 156.99$.
- We can use rational powers to approximate irrational powers.
- This is very similar reasoning to defining the movement π on the number line.

We can finally complete the table in *Summary: Defining Powers*.

Exponent type	Working definition of power & Why it makes sense
irrational	Let $a, x \in \mathbb{R}$ and x is irrational. Define a^x as $\lim_{n\to\infty} a^{x_n}$, where $x_0, x_1, x_2, \ldots$ is a sequence of rational numbers whose limit is x. *Example.* 5^π. See *Sequence of Powers*. Why: The function $f(x) = 5^x$ is continuous. This means when the change in inputs gets closer to 0, the change in outputs also gets closer to 0. *Example.* As $\Delta x, \Delta y \to 0$, the output difference $\lvert 5^x - 5^y \rvert \to 0$.

We did it! We defined powers for all real exponents.

Our journey to do so involved:

- Defining Numbers: We thought of numbers as movement along the number line. Using this model, we defined positive integers, negative integers, unit fractions, all rational numbers. Finally, we defined all irrational numbers.
- Power Properties: We identified three power properties, the First Power Property, Product of Powers, and Power of Powers.

- Defining Properties: We explored the consequences of the three power properties and found ways to define negative integer powers, unit fraction powers, and rational powers. The reasoning for these properties were parallel to how numbers are defined. The main difference is that when defining numbers, we think additively, and when defining powers, we think multiplicatively. And for irrational powers, we use sequences of rational numbers.

In the next lesson, we will take this one step further and explore more consequences of how we have defined rational and irrational powers.

2 Do Our Definitions Work? Trouble with Negatives and Zero (Lesson 2) (Length: ~2 hours)

Re-examining how we constructed powers

Opening Inquiry: Decisions have consequences

0. (a) What are the three power properties? (b) What is the definition of $a^{1/q}$? (c) What is the definition of $a^{p/q}$? (Look these up from your notes if you can't remember.)
1. What is $(-8)^{1/3}$?
2. Find three ways to simplify the expression $(-8)^{2/6}$.

Takeaways, in the form of a concern that turns our exponential world upside down:

- Does our definition of rational powers actually work? Because it seems we found that sometimes $(a^2)^{1/6} \neq (a^{1/6})^2$, and that there might be other examples of this. Is it just negative numbers that are a problem?
- Does this mean the power properties might not be true? In particular, the example above breaks the power of power property (sometimes $(a^{1/6})^2 \neq a^{1/3}$).

Takeaways, for what to do about this:

- First, let's observe that decisions have consequences. We chose to define powers in a certain way, and to choose to believe in certain properties. The above contradictions are a consequence of these decisions.
- We are going to look back at how we derived our working definitions of powers. We will analyze the assumptions we would need for all statements to always be true, so that we can so that we can figure out the domains on which the various properties hold – or do not hold.

Let's re-examine how we constructed powers. Look at your sheet of definitions. We'll take one exponent case at a time.

Positive integer powers

Positive integer powers: Are products and powers well-defined?

1. How is a^x defined when $x \in \mathbb{N}_{>0}$?
2. Let the exponents x_1, x_2 be positive integers.
Are products and powers of powers well-defined for any value of the base $a \in \mathbb{R}$?
Complete the table. Mark ✓ for "Yes" and × for "No".

	Are these equalities always well-defined and true?	
When $x_1, x_2 \in \mathbb{N}_{>0}$ and ↓	$a^{x_1}a^{x_2} = a^{x_2}a^{x_1}$	$(a^{x_1})^{x_2} = (a^{x_2})^{x_1}$
$a > 0$		
$a = 0$		
$a < 0$		

3. Reasoning:

Positive integer exponents means we can use elementary school ideas of multiplication and addition, and counting dots.

We'll do Powers of Powers here and you'll have Product of Powers for homework.

Product of Powers: Homework.

Power of Powers: No matter what a is, we always have:

$$\boxed{(a^{x_1})^{x_2}} = (\underbrace{a \cdot a \cdots a}_{x_1})^{x_2} \quad \text{(defn of positive integer power, exponent } x_1\text{)}$$

$$= \underbrace{\underbrace{a \cdot a \cdots a}_{x_1} \cdot \underbrace{a \cdot a \cdots a}_{x_1} \cdots \underbrace{a \cdot a \cdots a}_{x_1}}_{x_2} \quad \text{(defn of positive integer power, exponent } x_2\text{) (*)}$$

We can visualize this product as:

$$x_2 \left\{ \begin{array}{l} a \cdot a \cdots a \\ a \cdot a \cdots a \\ \quad \vdots \\ \underbrace{a \cdot a \cdots a}_{x_1} \end{array} \right. \qquad \text{or} \qquad \underbrace{x_1 \left\{ \begin{array}{cccc} a & a & & a \\ a & a & & a \\ \vdots & \vdots & \cdots & \vdots \\ a & a & & a \end{array} \right.}_{x_2}$$

Hence, (*) equals:

$$= \underbrace{\underbrace{a \cdot a \cdots a}_{x_1} \cdot \underbrace{a \cdot a \cdots a}_{x_1} \cdots \underbrace{a \cdot a \cdots a}_{x_2}}_{x_1} \quad \text{(commutativity of multiplication in } \mathbb{Z}\text{, applied to } x_1 x_2\text{)}$$

$$= \boxed{(a^{x_2})^{x_1}} \quad \text{(defn of positive integer power, exponent } x_1 x_2\text{)}$$

All of the expressions have real values, by closure of multiplication in $\mathbb{R}$.

Natural and Integer Powers

Adding and multiplying integers $\longleftrightarrow$ Products and powers of integer powers

1. How is a^x defined when $x = 0$ and $x \in \mathbb{Z}, x < 0$?
   When $x = 0$: When $x < 0$:
2. Here are some sample solutions to homework problems. What questions and comments do you have?
   - *Explain why* $-5 + 3 = 3 + -5$:
     - Think of -5 as five movements by -1, and $+3$ by three movements of $+1$.
     - The LHS (left hand side expression) is like starting at 0, moving -5, then moving $+3$.
       The RHS (right hand side expression) is like starting at 0, moving $+3$, then moving -5.
     - Either way, is the net result of moving -1 five times and $+1$ three times.
     - Movements of -1 paired with movements of 1 result in a net movement of 0, because -1 and $+1$ are additive inverses.
     - After pairing up the movements, there are two $+1$ movements remaining.
     - So $-5 + 3 = 3 + -5 = +2$.
   - *Explain why* $(-3) \times (-2) = 6$:
     - Multiplying -3 by -2 means taking the movement -3 twice and then switching the direction of the result.
     - Twice the movement -3 is the movement -6.
     - Switching the direction of the movement -6 gives us the movement $+6$.
     - So $(-3) \times (-2) = 6$.
3. Suppose your students have just learned that a^{-1} means $\frac{1}{a}$ and they are still struggling with simplifying fractional expressions with variables, as well as remembering what a^{-1} means. With the above in mind, work on (a)(b)(c).
   In all your explanations, be sure to:
   - Use the definition of a^{-1}
   - Use the definition of a^{-n}, where $n \in \mathbb{N}$
   - Use a structure as similar as possible to one of the explanations above.

   (a) What is an explanation that could help your students understand why $a^{-5}a^3 = a^3a^{-5}$?
   (b) What are $a^n a^{-m}$ and $a^{-m}a^n$ when $n > m$? When $n = m$? When $n < m$?
   (c) What is an explanation that could help your students understand why $(a^{-3})^{-2} = a^6$?

Natural and integer powers: Do power properties hold?

Let $x_1, x_2 \in \mathbb{Z}$.

Are the power properties well-defined for any value of the base $a \in \mathbb{R}$?

Complete the table. Mark ✓ for "Yes" and × for "No".

When $x_1, x_2 \in \mathbb{Z}$ and ↓	(a) Are these equalities always well-defined and true? (b) What number line movement and location ideas do these most resemble? $a^{x_1}a^{x_2} = a^{x_2}a^{x_1}$	$(a^{x_1})^{x_2} = (a^{x_2})^{x_1}$
$a > 0$		
$a = 0$		
$a < 0$		

Reasoning:

Summary:

- For negative numbers $-n$, the power a^{-n} is defined as $(a^{-1})^n = (\frac{1}{a})^n$.
- The expression $\frac{1}{a}$ is not well-defined when $a = 0$.
- Let $n, m \in \mathbb{N}$. We can break down products of powers into four cases: $a^n a^m, a^{-n}a^{-m}, a^n a^{-m}, a^{-n}a^m$.
  - Case 1. Already done (positive integer powers).
  - Case 2. $a^{-n}a^{-m} = (\frac{1}{a})^n(\frac{1}{a})^m$. Because n, m are positive integers, we know $(\frac{1}{a})^n(\frac{1}{a})^m = (\frac{1}{a})^m(\frac{1}{a})^m = a^{-m}a^{-n}$.
  - Case 3 and 4. See Question #3 in *Adding and multiplying integers ⟷ Products and powers of integer powers*. In all cases, $a^n a^{-m} = a^{-m}a^n = a^{n-m}$. So because it doesn't matter whether the first power is a negative power or positive power, Case 4 follows from Case 3.
- Let $n, m \in \mathbb{N}$. We can break down powers of powers similarly to products of powers, but there may be more work to be done for Case 3 and Case 4. This is for homework.

RATIONAL POWERS

Divisions and fractions $\longleftrightarrow$ Rational powers

1. How is $a^{\frac{p}{q}}$ defined, for $p, q \in \mathbb{Z}, q \neq 0$?
   $p > 0$: $p = 0$: $p < 0$:
2. Here are some sample solutions to homework problems. What questions and comments do you have?
   - *There is only one number that can be the result of* $5 \div 3$. (*)
     - Let $x = 5 \div 3$. By definition of division, x is the quantity such that $x \times 3 = 5$.
     - Let $y \in \mathbb{R}$. Either $y = x$, $y > x$, or $y < x$.
     - Suppose $y < x$. Then $3y < 3x = 5$, so $y \neq 5 \div 3$.
     - Suppose $y > x$ Then $3y > 3x = 5$, so $y \neq 5 \div 3$.
     - Therefore $y = x$ and there is only one value such that $x = 5 \div 3$.
   - $5 \div 3$ *is the same number as* $\frac{1}{3} \cdot 5$.
     - Start with $x' = \frac{1}{3} \cdot 5$.
     - See what $3x'$ equals. If $3x' = 5$, then by (*), we can conclude $x' = x$.

$$\begin{aligned} 3x' = \frac{1}{3} \cdot 5 \cdot 3 &= \frac{1}{3} \cdot 3 \cdot 5 \quad \text{commutativity of multiplication in } \mathbb{Z} \\ &= 1 \cdot 5 \quad \text{definition of fraction} \\ &= 5 \quad \text{multiplicative identity} \end{aligned}$$

     - Hence $5 \div 3 = \frac{1}{3} \cdot 5$.
3. Suppose your students have just learned q-th roots, and they are still struggling with simplifying radical expressions and remembering what q-th roots are. They also need to review the definition of a^{-1} as $1/a$.
   With the above in mind, work on (a)(b)(c).
   In all your explanations, be sure to:
   - Use the definition of $a^{1/q}$
   - Use the definition of $a^{p/q} = (a^{1/q})^p$
   - Use the definitions of a^{-1} and a^{-n}, for $n \in \mathbb{N}$
   - Use a structure as similar as possible to the explanations above.

   (a) Help your students understand why $(7^5)^{1/3} = (7^{1/3})^5$.
   (b) Help your students understand why, in general, $(a^p)^{1/q} = (a^{1/q})^p$. (Assume a, p, q are all positive and $p, q \in \mathbb{N}$.)
   (c) Help your students understand why $(7^{-5})^{1/3} = (7^{1/3})^{-5}$.

Rational powers of negative bases

1. Analyze this claim: $(-1)^{\frac{1}{3}}(-1)^{\frac{1}{9}} = (-1)^{\frac{1}{3}+\frac{1}{9}}$. Is the equation a true statement? A false statement?
2. Now analyze this claim: $((-7)^5)^{1/3} = ((-7)^{1/3})^5$.
3. Using only the numbers $\pm 1, \pm 2, \pm 4, \pm 8$, come up with new examples that break the power of powers property.
4. Based on the definition of $a^{1/q}$ and $a^{p/q}$, does $((-7)^5)^{1/3} = ((-7)^{1/3})^5$? $((-7)^4)^{1/3} = ((-7)^{1/3})^4$? $((-7)^3)^{1/4} = ((-7)^{1/4})^3$? $((-7)^{10})^{1/4} = ((-7)^{1/4})^10$? For what p and q do these definitions seem to work for negative bases without problems?

Rational powers: Do power properties hold?

Let the exponents x_1, x_2 be rational.

Are the power properties well-defined for any value of the base $a \in \mathbb{R}$?

Complete the table. Mark ✓ for "Yes" and × for "No".

When x_1, x_2 are zero or negative and ↓	(a) Are these equalities always well-defined and true? (b) What fraction ideas do these most resemble? $a^{x_1}a^{x_2} = a^{x_2}a^{x_1}$	$(a^{x_1})^{x_2} = (a^{x_2})^{x_1}$
$a > 0$		
$a = 0$		
$a < 0$		

Reasoning:

Summary:

- Product of powers and powers of powers are not well-defined when $a = 0$ (same reasons as before)
- Product of powers and powers of powers are not well-defined for $a < 0$ when we are restricted to real numbers.
- This is because $a^{1/q}$ is not real when q is even.
- This is the issue that came up in our opening inquiry. We can use complex numbers to resolve this issue. However, at the high school level, prior to learning complex numbers, the opening inquiry shows why fractional powers of negative numbers seem to bring up contradictions.

Real powers

Limits in high school curricula

For homework, you investigated:

Here are some places where limits and convergence can appear in high school curricula:

- Geometry: *Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri?s principle, and informal limit arguments.* (Common Core G-GMD.1)
- Pre-calculus: *Limits of functions and sequences.* (Lincoln Public School standards and course descriptions, `https://home.lps.org/math/secondary/#`)
- AP Calculus: *The derivative of a function is defined as the limit of a difference quotient and can be determined using a variety of strategies. (College Board Course and Exam Description: AP Calculus AB and AP Calculus BC, `https://apcentral.collegeboard.org/pdf/ap-calculus-ab-and-bc-course-and-exam-description.pdf`)*

(a) Describe an example of a convergent sequence that you might use while teaching one of geometry, pre-calc, or calculus.

(b) What is an (informal) explanation of what it means for a sequence to converge or limit to something? This explanation should work across all these courses.

Here is a possible response. What questions and comments do you have?

(a)
- *Geometry.* Given a circle with radius r. Let c_1 be the area of an equilateral triangle inscribed in the circle. Let c_2 be the area of a square inscribed in the circle. Let c_3 be the area of a regular pentagon inscribed in the circle. Keep on increasing the number of sides of the inscribed polygon in the circle, so c_n is the area of a regular $n+2$-gon inscribed in the circle. The limit of the sequence $\{c_1, c_2, \ldots\}$ is the area of the circle. That is, $\lim_{n\to\infty} c_n = \pi r^2$.
- *Pre-calculus.* Let $c_1 = \frac{1}{2}, c_2 = \frac{3}{4}, c_3 = \frac{7}{8} \ldots, c_n = 1 - \frac{1}{2^n}, \ldots$. Then the sequence $\{c_1, c_2, \ldots\}$ converges to 1. That is, $\lim_{n\to\infty} c_n = 1$.

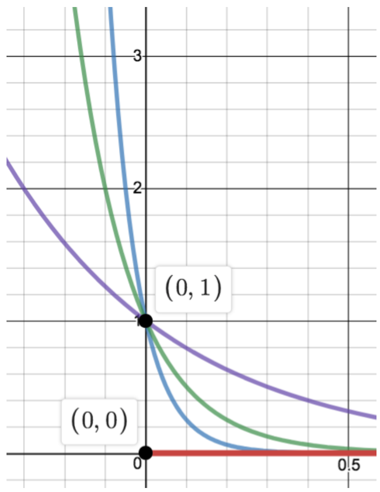
- *AP Calculus.* The instantaneous rate of change of a function f at a point x_0 is $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$. Let $c_1 = \frac{f(x_0+0.1)-f(x_0)}{0.1}$, $c_2 = \frac{f(x_0+0.01)-f(x_0)}{0.01}$, $\ldots c_n = \frac{f(x_0+10^{-n})-f(x_0)}{10^{-n}}$, $\ldots$. When this sequence converges, we have $f'(x_0) = \lim_{n\to\infty} c_n$.

(b) *A sequence* $\{c_0, c_1, c_2, \ldots\}$ *converges to a real number if, as n gets larger,* c_n *and* c_{n-1} *get closer and closer to each other to the point where there is practically 0 difference between them.*

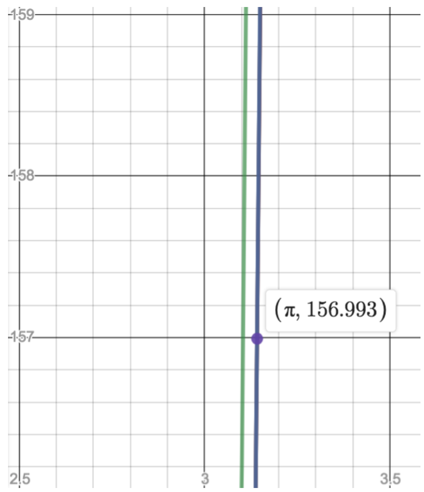
Limits of powers

For homework, you graphed variations of 0^x and 5^x.

The graphs you sketched looked like:



$0^x, (0.1)^x, (0.001)^x (0.000001)^x$
near $x = 0$



$5^x, (5.1)^x, (5.001)^x (5.000001)^x$
near $x = \pi$

(a) Look at the first set of variations. As we move from the graph of 0.1^x to 0.001^x to 0.000001^x, what happens to the instantaneous slope at the point $(0, 1)$?

(b) What do you think the slope of $\lim_{n\to\infty} \left(\frac{1}{10^n}\right)^x$ would be, near $x = 0$?

(c) If the function 0^x could be assigned a slope at $x = 0$, what should it be?

(d) How do you think the slope of $\lim_{n\to\infty} \left(5 + \frac{1}{10^n}\right)^x$ compares to the slope of 5^x, near $x = \pi$?

Takeaways:

Real powers: Do power properties hold?

1. How is a^x defined when $x \in \mathbb{R}$?

2. Let the exponents $x_1, x_2 \in \mathbb{R}$.
   Are the power properties well-defined for any value of the base $a \in \mathbb{R}$?
   Complete the table. Mark ✓ for "Yes" and × for "No".

When x_1, x_2 are zero or negative and ↓	Are these equalities always well-defined and true? $a^{x_1}a^{x_2} = a^{x_2}a^{x_1}$	$(a^{x_1})^{x_2} = (a^{x_2})^{x_1}$
$a > 0$		
$a = 0$		
$a < 0$		

3. Reasoning:

Summary:

- Irrational powers are limits of rational powers, so if sequences of rational powers may not be well-defined, the limit of such a sequence will also be not well-defined.
- The exponential function is "nice" as long as $a > 0$ and $x \neq 0$, as shown in the *Limits of Powers* tasks. This means (thanks to ideas from calculus/real analysis) that we can use limits to define what it means to take products and powers of powers.

Power Properties: Final Version for Real Bases

We began this lesson showing that power properties run into problems.

We then went back over our definitions and examined, for each one, when the power properties would be okay and when they would not be. We found:

Property 2.1 (Power Properties)**.** Let $a \in \mathbb{R}, a > 0$ and $x, x_1, x_2 \in \mathbb{R}$. Then exponential expression of the form a^x satisfy the following properties:

- (1st power) $a^1 = a$
- (product of powers property) $a^{x_1+x_2} = a^{x_2+x_1} = a^{x_1}a^{x_2}$
- (power of a power property) $(a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1x_2}$

Note: Power properties allow us to say that, for positive bases, exponent addition and multiplication are commutative, associative, and satisfy the distributive property.

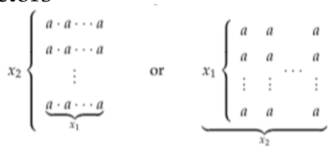
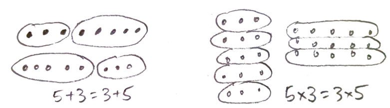
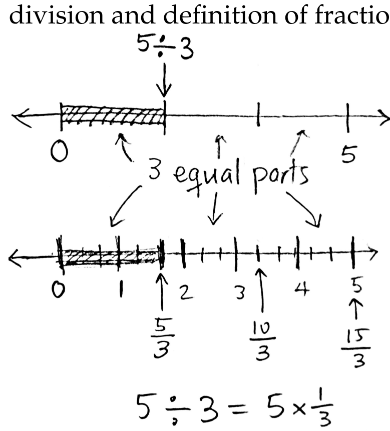
When $a \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{N}_{>0}$, the power properties hold. When $a \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{Z}$, the power properties still hold as long as $a \neq 0$. But when $x_1, x_2 \in \mathbb{Q}$ or $x_1, x_2 \in \mathbb{R}$, then we need to restrict the base to the domain $a > 0$.

Explaining Power Properties: Parallels in K-12 Mathematics

In this process, we saw yet more parallels between how kids grapple with number and operation and how middle school and high school students might grapple with exponentiation. We also saw how ideas in exponentiation can support learning pre-calculus and calculus.

The table below summarizes these parallels.

Through all this, we were able to connect different parts of the curriculum to see a bigger story, as well as see how explaining power properties draws on and builds toward concepts in K-12. In your homework, you will have an opportunity to practice using these parallels.

Explaining power properties	Parallel in K-12
Power properties for positive integer exponents How: Counting positive integer number of real factors  	Commutativity of addition and multiplication of positive integers How: Counting dots 
Power properties for integer exponents How: Multiplying and dividing factors; pair up multiplicative inverses.	Commutativity of addition and multiplication of integers How: Location/movement on number line; pair up additive inverses
Power properties for rational non-integer exponents How: Compare powers; use definition of q-th root, which is based on definition of fraction	Arithmetic properties of division and fractions (e.g., $p \div q = \frac{1}{q} \cdot p$) How: Comparing lengths; use definition of division and definition of fraction 
Power properties for real non-rational exponents How: Use sequences of rational powers; use meaning of convergent sequence for values and rates of change	Limits of sequences, in pre-calculus and calculus How: Meaning of convergence

Appendix to Chapter: Proofs that Power Properties Extend

Extending to Natural Powers

> Spoiler alert: The below conjecture is *almost* true, and the proof *almost* works.
> To make it true, we need to restrict the domain of bases to *nonzero* $a \in \mathbb{R}$.

Conjecture. For all $a \in \mathbb{R}$, natural powers of a satisfy the three Power Properties.

Proof approach? Given $a \in \mathbb{R}$.

- **1st power**. Defining a^0 does not impact the first property, which only deals with the power a^1. So this property holds.
- **Product of powers.** To show this property, we must show that $a^x a^y = a^y a^x = a^{x+y}$ is a true statement all pairs of natural x, y. There are four cases:
  - Case $x > 0, y > 0$. Then $a^x a^y = a^y a^x = a^{x+y}$ by the truth of Conjecture **??**.
  - Case $x = 0, y > 0$. Then

$$\begin{aligned} a^0 a^y &= 1 \cdot a^y \\ &= a^y \\ &= a^{0+y}. \end{aligned}$$

    Commutativity follows from commutativity of + and $\cdot$ in $\mathbb{Z}$.
  - Case $x > 0, y = 0$. (Try yourself. It is similar to the above.)
  - Case $x = 0, y = 0$. (Try yourself. It is similar to the above.)
- **Power of powers.** (Try yourself.)

□

> **Proof approach analysis: Natural number powers**
> This proof approach is so close to working! But it has some tiny flaws.
> Take a look at the Product of Powers steps. For equal signs, identify the reasons that the equality holds.
> Based on this, what flaws are there? Takeaway (to be filled in as class):

Takeaway: The proof does not work for $a = 0$. But it works for all other $a \in \mathbb{R}$. So we need to revise what we conjectured to reflect what we know:

Theorem 2.2. *For all* nonzero $a \in \mathbb{R}$, *natural powers of a satisfy the three Power Properties.*

Proof idea. Use the proof for above conjecture, restricting to $a \neq 0$. □

Extending to Integer Powers

Theorem 2.3. *For all* nonzero $a \in \mathbb{R}$, *integer powers of a satisfy the three Power Properties.*

Sketch of proof. Given $a \in \mathbb{R}, a \neq 0$.

- **1st power**. This property holds because it it not impacted by negative powers.
- **Product of powers.** To show this property, we must show that $a^x a^y = a^y a^x = a^{x+y}$ is a true statement all pairs of integer x, y. There are four cases:

- Case $x \geq 0, y \geq 0$. Then $a^x a^y = a^y a^x = a^{x+y}$ by Theorem 2.2.
- Case $x < 0, y \geq 0$. Let $x = -n$, where $n \in \mathbb{N}$.

$$\begin{aligned} a^x a^y &= a^{-n} a^y \quad \text{(definition of } n\text{)} \\ &= (\frac{1}{a})^n \cdot a^y \quad \text{(definition of negative integer power)} \\ &= a^{y-n} \quad (\frac{1}{a} \text{ is the multiplicative inverse of } a) \\ &= a^{x+y} \quad \text{(definition of } n\text{)} \end{aligned}$$

- Case $x \geq 0, y < 0$. Similar to the above.
- Case $x < 0, y < 0$. Let $x = -n, y = -m$, where $n, m \in \mathbb{N}$.

$$\begin{aligned} a^x a^y &= a^{-n} a^{-m} \quad \text{(definition of } n, m\text{)} \\ &= \left(\frac{1}{a}\right)^n \cdot \left(\frac{1}{a}\right)^m \quad \text{(definition of negative integer power)} \\ &= \left(\frac{1}{a}\right)^{n+m} \quad \text{(definition of positive integer power)} \\ &= a^{-n-m} \quad \text{(definition of negative integer power)} \\ &= a^{x+y} \quad \text{(definition of } n, m\text{)} \end{aligned}$$

- **Power of powers.**
  - Case $x \geq 0, y \geq 0$. Then $(a^x)^y = (a^y)^x = a^{xy}$ by Theorem 2.2.
  - Case $x < 0, y \geq 0$. Let $x = -n$, where $n \in \mathbb{N}$. Then

$$(a^x)^y = (a^{-n})^y = \left(\left(\frac{1}{a}\right)^n\right)^y = \left(\frac{1}{a}\right)^{ny} = a^{-ny} = a^{xy}.$$

  - Case $x \geq 0, y < 0$. Similar to above.
  - Case $x < 0, y < 0$. Let $x = -n, y = -m$, where $n, m \in \mathbb{N}$. Then

$$(a^x)^y = (a^{-n})^{-m} = \left(\left(\frac{1}{a}\right)^n\right)^{-m} = \left(\frac{1}{\left(\frac{1}{a}\right)^n}\right)^m = (a^n)^m = a^{nm}.$$

  Because $nm = (-n)(-m)$, we can conclude $(a^x)^y = a^{xy}$.

Commutativity for power of powers and product of powers properties follows from commutativity of + and $\cdot$ in $\mathbb{Z}$. □

Extending to Rational Powers

Spoiler alert: The conjecture is false! To be true, we need to restrict the base a to positive reals, or do a very finicky restriction of a based on whether q is odd or even.

What is the definition of q-th root?

What is the definition of $a^{p/q}$?

Before getting into the conjecture proof, we go over the lemma that is key to showing this conjecture which you saw for homework.

Lemma for rational power properties

Read the statement of this theorem:

Theorem 1.6. *For all positive $a \in \mathbb{R}$ and positive $q \in \mathbb{Z}$, there exists a unique positive q-th root of a.*

The word "positive" is used three times, to refer to different things. Using the notation $a^{1/q}$, and the terms "exponent", "base", and "power", explain what each "positive" is talking about:

- The first "positive" is talking about ______________________.
- The second"positive" is talking about ______________________.
- The third "positive" is talking about ______________________.

Read over the proof of the Lemma below. Why does Theorem 1.6 help prove the lemma? How would you explain the logic of the proof?

Lemma. *For all $a \in \mathbb{R}, p, q \in \mathbb{Z}, q \neq 0$, if $\sqrt[q]{a}$ and $\sqrt[q]{a^p}$ exist, then*

$$\sqrt[q]{a^p} = (\sqrt[q]{a})^p.$$

Proof of Lemma. Given $a \in \mathbb{R}, p, q \in \mathbb{Z}, q \neq 0$, and $\sqrt[q]{a}$ and $\sqrt[q]{a^p}$ exist.

$$\begin{aligned} ((\sqrt[q]{a})^p)^q &= (\sqrt[q]{a})^{pq} && \text{(power of a power works for all natural powers)} \\ &= (\sqrt[q]{a})^{qp} && \text{(power of a power for natural powers)} \\ &= ((\sqrt[q]{a})^q)^p && \text{(power of a power for natural powers)} \\ &= a^p && \text{(definition of } q\text{-th root).} \end{aligned}$$

We have shown $(\sqrt[q]{a})^p$ is a q-th root of a^p, so by Theorem 1.6, $\sqrt[q]{a^p} = (\sqrt[q]{a})^p$. □

Explanation of why Theorem 1.6 helps make the proof work:

Conjecture. For all nonzero $a \in \mathbb{R}$, rational powers of a satisfy the three Power Properties.

Proof approach? Given $a \in \mathbb{R}, a \neq 0$.

- **1st power**. We show: If $p/q = 1$, then the definition of rational powers implies $a^{p/q} = 1$.
  Suppose $p/q = 1$. Then $p = q$.

$$\begin{aligned} a^{p/q} = a^{p/p} &= (\sqrt[p]{a})^p && \text{(definition of } a^{p/q}\text{)} \\ &= a && \text{(definition of } p\text{-th root)} \\ &= a^1 && \text{(1st power)} \end{aligned}$$

- For showing product and power properties, let $p, q, r, s \in \mathbb{Z}$, where $q, s \neq 0$.
- **Product of powers.** We show: $a^{p/q}a^{r/s} = a^{\frac{ps+qr}{qs}}$. $(*)$
  Using our lemma, let us write:
  - the left hand side as $LHS = bc$, where $b = \sqrt[q]{a^p}$ and $c\sqrt[s]{a^r}$.
  - the right hand side as $RHS = \sqrt[qs]{a^{ps+qr}}$.

To show the equality $(*)$, we examine $(LHS)^{qs}$:

$$\begin{aligned}(LHS)^{qs} &= (bc)^{qs} && \text{definition of } LHS \\ &= b^{qs}c^{qs} && \text{definition of exponentiation for integer } qs\text{, and commutativity of } \cdot \text{ in } \mathbb{R} \\ &= (b^q)^s(c^s)^q && \text{power of powers for integers } q, s \\ &= (a^p)^s(a^r)^q && \text{definition of } b, c\text{; definition of } q\text{-th root, } s\text{-th root} \\ &= a^{ps}a^{rq} && \text{power of powers for integers } p, q, r, s \\ &= a^{ps+rq} && \text{product of powers for integers } ps, rq.\end{aligned}$$

By Theorem 1.6, $bc = \sqrt[qs]{a^{ps+rq}}$, meaning $a^{p/q}a^{r/s} = a^{\frac{ps+qr}{qs}}$, as desired.

Power of powers. We show: $(a^{p/q})^{r/s} = a^{\frac{pr}{qs}}$. $(**)$

Using our lemma, let us write:

- the left hand side as $LHS = (b^r)^{1/s}$, where $b = \sqrt[q]{a^p}$
- the right hand side as $RHS = \sqrt[qs]{a^{pr}}$.

To show the equality $(**)$, we examine $(LHS)^{qs}$:

$$\begin{aligned}(LHS)^{qs} &= ((b^r)^{1/s})^{qs} && \text{definition of } LHS \\ &= (((b^r)^{1/s})^s)^q && \text{power of powers for integers } q, s \\ &= (b^r)^q && \text{definition of } s\text{-th root} \\ &= (b^q)^r && \text{power of powers for integers } q, r \\ &= (a^p)^r && \text{definition of } b \text{ and } q\text{-th root of } a^p \\ &= a^{pr} && \text{power of powers for integers } q, r.\end{aligned}$$

By Theorem 1.6, $LHS = \sqrt[qs]{a^{pr}}$, meaning $(a^{p/q})^{r/s} = a^{\frac{pr}{qs}}$ as desired. □

Proof approach analysis: Rational number powers

This proof approach is very promising, but it also has flaws.

One of the key pieces of the approach is using the Lemma:

> **Lemma.** *For all $a \in \mathbb{R}$, $p, q \in \mathbb{Z}$, $q \neq 0$, if $\sqrt[q]{a}$ and $\sqrt[q]{a^p}$ exist, then $\sqrt[q]{a^p} = (\sqrt[q]{a})^p$.*

But $\sqrt[q]{a}$ and $\sqrt[q]{a^p}$ do not exist for all nonzero $a \in \mathbb{R}$. What a do they always exist for? What a do they sometimes exist for? What a do they never exist for?

Takeaway: (to be filled in as class)

Takeaway: We need to restrict the a in the statement of the conjecture. The simplest way to restrict it is to have positive a, and that is what middle school and high school textbooks often do.

Theorem 2.4. *For all* positive *$a \in \mathbb{R}$, rational powers of a satisfy the three Power Properties.*

Extending to Real Powers

We are very close to our finish line! There is one more thing to do – see if the properties extend to all real powers, including irrational powers!

Once again, when working with irrational numbers, approximations come to the rescue. Through approximations, what holds for the rationals also holds for the reals.

Theorem 2.5. *For all* positive *$a \in \mathbb{R}$, real powers of a satisfy the three Power Properties.*

Sketch of proof. Given $a \in \mathbb{R}, a > 0$.

- **1st power**. Since 1 is rational, $a^1 = 1$ is not impacted by extending to irrational powers.
- **Product of powers.** Let $x, y \in \mathbb{R}$. Let $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ be sequences of rational numbers such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. (If $x, y \in \mathbb{Q}$, then take constant sequences.)

  Then $a^x a^y$ can be approximated by $a^{x_n} a^{y_n}$, and a^{x+y} can be approximated by $a^{x_n+y_n}$, by continuity of exponentiation. In other words, $\lim_{n\to\infty} a^{x_n} a^{y_n} = a^x a^y$ and $\lim_{n\to\infty} a^{x_n+y_n} = a^{x+y}$. By product of powers of rational exponents, $a^{x_n} a^{y_n} = a^{x_n+y_n}$ for all n. Thus $a^x a^y = \lim_{n\to\infty} a^{x_n} a^{y_n} = \lim_{n\to\infty} a^{x_n+y_n} = a^{x+y}$.
- **Power of powers.** Similar reasoning to product of powers. Let $x, y \in \mathbb{R}$. Let $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ be sequences of rational numbers as above.

  Then $(a^x)^y$ can be approximated by $(a^{x_n})^{y_n}$, meaning and a^{xy} can be approximated by $a^{x_n y_n}$, by continuity of exponentiation. By powers of powers of rational exponents, $(a^{x_n})^{y_n} = a^{x_n y_n}$ for all n. Thus $(a^x)^y = \lim_{n\to\infty} (a^{x_n})^{y_n} = \lim_{n\to\infty} a^{x_n y_n} = a^{xy}$. □

3 Exploring Powers, Introducing Logarithms (Lesson 3, Part 1) (Length: ~1 hour)

Working with Powers from a Correspondence Perspective

The Power of One (and Zero)

Find all pairs of a, b such that $a^b = 1$.

Find all pairs of a, b such that $a^b = 0$.

We will take this as true:

Proposition 3.1. *Given $a, b \in \mathbb{R}, a > 0$.*

- $a^b = 1$ *if and only if* $a = 1$ *or* $b = 0$.
- $a^b = 0$ *if and only if* $a = 0, b \neq 0$.

Fun with powers

Find the solutions.

1. $\frac{5^{x^2}}{5^{4x}5^2} = 5^3$
2. $2^{(a^2-9)} = 1$
3. $3^{(4^{(5^x-4)}-3)} = 3$.
4. $a^{a(a^2-4)} = 1$.

Setting Up Existence of Logarithms

Are all numbers powers of a given number?

1. Using the calculator here, https://www.desmos.com/scientific, estimate x such that $5^x = 74$, so that you are within the first two decimal places of the exact value of x.
2. How do you know that the first 2 decimal digits are correct?
3. Let P be a random positive real number. Is it always possible to find an $x \in \mathbb{R}$ such that $5^x = P$? Why or why not?

Takeaway:

Takeaway: Yes, given any $a \neq 1$, any positive real number $a > 0$ is a power of a.

Proposition 3.2. *Let $P \in \mathbb{R}_{>0}$. For all $a \in \mathbb{R}_{>0}, a \neq 1$, there exists an $x \in \mathbb{R}$ such that $a^x = P$.*

Sketch of proof. There are two cases: $a = 1, a \neq 1$.

- When $a = 1$, all powers of a are 1, so only $P = 1$ is a power of a.
- When $a \neq 1$, we can get closer and closer to finding the x such that $a^x = P$ by increasing the number of decimal digits.

  This makes sense because changing the exponent by a little bit should only change the power by a little bit, and the further out the decimal digit, the less we change the exponent by writing another digit.

□

Finding the Exponent

Here are two approximations:

$$5^{1.37} = 9, \quad 5^{0.43} = 2.$$

Using these values, we can make other approximations. For example:

Power 5^x	Exponent x	Reasoning
$5^{\boxed{?}} = 81$	$\boxed{?} = 2.74$	$81 = 9^2$
		$= 5^{1.37} \cdot 5^{1.37}$ (given)
		$= 5^{2.74}$ (product of power property)
		$\boxed{?} = 2.74$

Using the information below, and without a calculator, complete the following table.

Power 5^x	Exponent x	Reasoning
$5^x = 8$		
$5^x = 18$		
$5^x = 45$		
$5^x = 10$		
$5^x = 100$		
$5^x = 900$		
	$3 + 0.43$	
	$\frac{1}{4} \cdot 2.74$	
	-2.86	

Exponential functions: invertible?

Let $f(x) = 5^x$. Fill in sample entries of an input/output table to represent the inverse of f.

f^{-1}

input	output

Is f invertible? Why or why not?

If $g(x) = a^x$, and $a > 0$, is g invertible?

Proposition (INCORRECT). Let $a > 0$, and $f(x) = a^x$. Then f is invertible.

INCORRECT proof. Given $a > 0$, $f(x) = a^x$.

A function f is invertible if its inverse is a function, meaning:

- Every power can only come from one exponent.
- If $y_0 = a^{x_0}$, then there is no other x such that $y_0 = a^x$.

Suppose $a^x = a^{x_0}$. Then $a^{x - x_0} = 1$.

So $x - x_0 = 0$, and $x = x_0$. Hence f is invertible. □

What's wrong with the proof? We just said 1^x is invertible, which it very much isn't.

Proposition 3.3 (CORRECT). *Let $f(x) = a^x$, where $a > 0$. If $a \neq 1$, then f is invertible. If $a = 1$, then f is not invertible.*

Correct proof. Given $a > 0$, $f(x) = a^x$.
f is invertible if its inverse is a function, meaning:

- every power can only come from one exponent.
- If $y_0 = a^{x_0}$, then there is no other x such that $y_0 = a^x$.

Case $a = 1$. Then all powers of a^x are 1, so f cannot be invertible.

Case $a \neq 1$. Suppose $a^x = a^{x_0}$. Then $a^{x-x_0} = 1$.
When $a \neq 1$, $a^x = 1$ if and only $x = 0$.
So $x - x_0 = 0$, and $x = x_0$. Hence f is invertible. □

Logarithm Definition and Properties

Definition 3.4 (Logarithm). For all positive real a, P, $a \neq 1$, we say the **logarithm base a of P** is the power of a that equals P.
We use $\log_a P$ to denote this value.

Example. $\log_5 9 = 1.37$, $\log_5 2 = 0.43$, $\log_5 81 = (1.37)^2$, $\log_5 25 = 2$, $\log_5 100 = 2.86$

Note: If $f(x) = a^x$, and $a > 0, a \neq 1$, then f is invertible and $f^{-1}(x) = \log_a(x)$.

Generalizing how to Find Logs

Let $L = \log_5 9$ and $M = \log_5 2$.

1. Complete the given rows as shown.
2. Then find as many other powers of 5 as you can that you can write in terms of L and M. Record your findings in the table.

Power P	Representation in terms of known powers of 5	Use the representation to express $\log_5 P$
9	–	$\log_5 9 = L$
2	–	$\log_5 2 = M$
81	9^2	$\log_5 9^2 = 2L$
$\frac{9^3}{5^{20}}$		
$900^{\frac{3}{4}}$		
$\sqrt[3]{2}$		
$\frac{1}{\sqrt[6]{50}}$		
⋮	⋮	⋮

Theorem 3.5 (Logarithm properties). *For all $a \in \mathbb{R}, a \neq 1$ and $P, Q, r \in \mathbb{R}$, the logarithm base a satisfies:*

- (Log of Product rule) $\log_a(PQ) = \log_a P + \log_a Q$
- (Log of Powers rule) $\log_a(P^r) = r \cdot \log_a P$

Proof. Given $a \in \mathbb{R}, a > 0$ and $P, Q, r \in \mathbb{R}$.

Product rule. Homework.

Power rule. By Proposition 3.2, there is $L \in \mathbb{R}$ such that $a^L = P$. We can write $P^r = (a^L)^r$. By the Power of Power Property of exponents, $(a^L)^r = a^{rL}$. By definition of log base a, we have $\log_a P^r = \log_a(a^{rL}) = rL = r \log_a P$. We have shown the Log of Power Rule.

□

Part II

Covariation View on Exponentiation

4 Exponential Growth is Defined by Constant Change Factors (Lesson 3, Part 2) (Length: ~1 hour)

Warm Up: Introducing Constant Change Factors

Jactus growth

One way that exponential growth has been contextualized for students is through stories of Jactus plants. Here are some example tasks based on the SPARQ curriculum materials, led by Amy Ellis, who created the Jactus idea.

Plants-R-Us, a unique plant store, is asking you to investigate a new plant they have recently discovered, the Jactus plant. It grows faster than any plant they have ever seen!

Some gardeners at the plant factory measured the heights of different varieties after different weeks and gave us this data.

(a) **The Flowering Jactus**:

Weeks	0	1	2	3	4	9
Height (inches)	$\frac{1}{2}$	1	2	4	8	256

(b) **The Tropical Jactus**:

Time passed (Weeks)	*Height (Inches)*
4	25.6
6	409.6
8	6553.6
10	104857.6
15	107374182.4
18	6871947673.6

The gardeners want us to help answer:

How fast do the plants grow?

Explore how the plants change from one week to the next using operations you know, for instance, subtraction and division. What patterns do you notice?

?

The gardeners want an estimate of each plant's height at 12 weeks. What estimate would you give? How would you justify this estimate?

Note: From now on, unless otherwise stated, we assume that when working with covarying quantity, the second quantity can be expressed as a function of the first quantity. We will use function notation to express the second variable as a function of the first, for instance, if x and y covary, we will write $y(0)$ to denote the value of y when $x = 0$. Depending on when it makes the communication clearer or easier, we will sometimes also give this function another name, for instance, saying that $y(x)$ can be expressed as $f(x)$.

Definition 4.1 (Constant change factors property)**.** We say that two covarying variables have **constant change factors** in the second variable if equal changes in the first variable correspond to equal change ratios in the second variable. The factor is the ratio.

Definition 4.2. Given two quantities with constant change factors in the second quantity. We say the 1**-unit change factor** is the change factor in the second quantity when the first variable changes by $+1$ unit. More generally, we say that the n**-unit change factor** is the change factor in the second quantity when the first quantity changes by n unit, where $n \in \mathbb{R}$.

> ?
>
> What is the 1-day change factor for the Flowering Jactus?

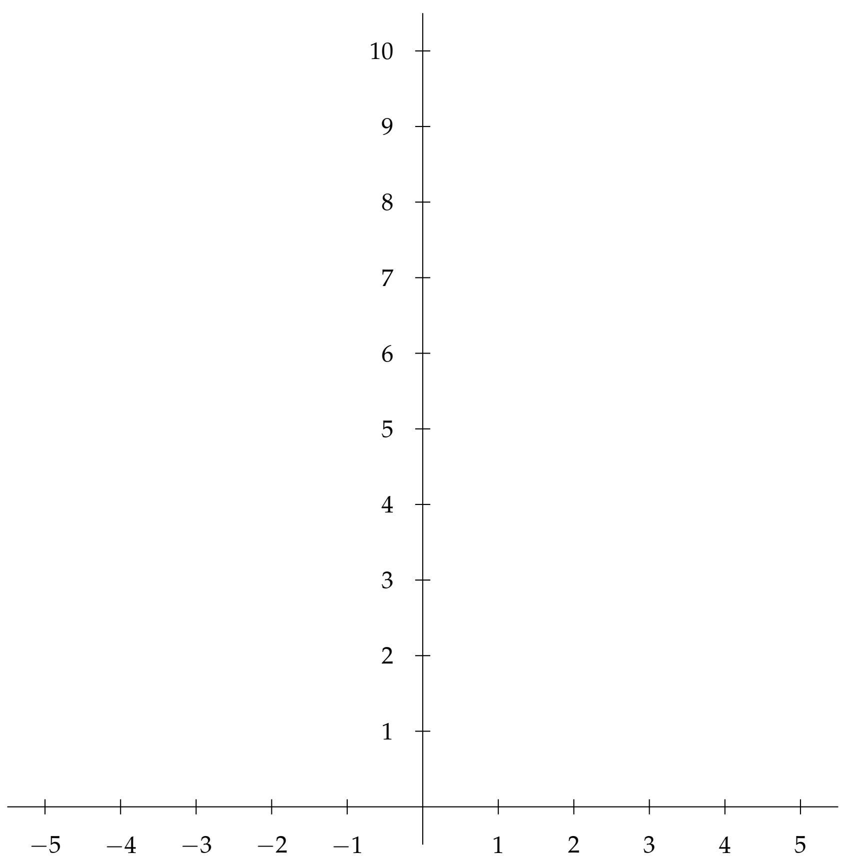
Main Inquiry: Implications of Constant Change Factors

Exploring the property of Constant Change Factors

A function $f : \mathbb{R} \to \mathbb{R}$ is such that $f(0) = 3$ and $f(1) = 3.75$.

Furthermore, f has Constant Change Factors in their output variables.

What are all possibilities for the graph of f? Plot them.



Equations of Functions with Constant Growth Factor

Find equations for the growth of the Flowering Jactus, the Tropical Jactus, and f from the task above.

The data for the Flowering Jactus and Tropical Jactus are given below for reference.

The Flowering Jactus:

Weeks	0	1	2	3	4	9
Height (inches)	$\frac{1}{2}$	1	2	4	8	256

The Tropical Jactus:

Time passed (Weeks)	*Height (Inches)*
4	25.6
6	409.6
8	6553.6
10	104857.6
15	107374182.4
18	6871947673.6

Constant Change Factors implies Exponential, and vice versa

Theorem 4.3 (Constant Change Factors if and only if Exponential)**.** *Suppose that two quantities x and y have constant change factors in y. Then there are unique $a, b \in \mathbb{R}$, $a \neq 0, b > 0$, such that $y = ab^x$.*

Moreover, if there are $a, b \in \mathbb{R}$, $a \neq 0, b > 0$, such that $y = ab^x$, then x and y have constant change factors in y, where a change of Δ in x corresponds to a change factor of b^Δ in y.

Sketch of proof.

[$\Rightarrow$] Given two quantities x and y with constant change factors in y.

Let $a = y(0)$, and suppose the 1-unit change factor is b. Then the only possibility for $y(1)$ is ab. There is only one solution to the equation $y(1)/y(0) = b$. Similarly, the only possibility for $y(2)$ is ab^2, and so on for all positive integers: $y(n) = ab^n$, where $n \in \mathbb{N}, n > 0$. And we can extend this reasoning to zero and negative integers inputs, and find the unique solution that $y(-n) = ab^{-n}$, where $n \in \mathbb{N}, n > 0$. Note that $y(0)$ can be expressed as ab^0, as $b > 0$ so $b^0 = 1$.

From here we can use $1/q$-unit change factors for every $q \in \mathbb{N}, q > 0$ to give a unique possibility for all rational number inputs, and find that for all $p, q \in \mathbb{Z}, q > 0$, we have $y(p/q) = x^{p/q}$.

We can then use limits of rational numbers to extend to all real inputs, and find that for all $\alpha \in \mathbb{R}$, we have $y(\alpha) = ab^\alpha$.

Hence for all real x, we have shown that there are unique $a, b \in \mathbb{R}$, $b > 0$, such that $y = ab^x$.

[$\Leftarrow$] Given $a, b \in \mathbb{R}$, $b > 0$, such that $y = ab^x$. Let Δ equal n units of the quantity x. We need to show that change by Δ in x results in a constant change factor in y. This amounts to showing that for all x, the quantity $y(x + \Delta)/y(x)$ is constant - in other words, it is independent of x.

$$\begin{aligned} \text{For all } x\text{ , } \frac{y(x+\Delta)}{y(x)} &= \frac{ab^{x+\Delta}}{ab^x} \quad \text{by given definition of } y \text{ and } x \\ &= \frac{ab^x b^\Delta}{ab^x} \quad \text{by exponent addition property for positive bases} \\ &= b^\Delta \quad \text{by exponent positivity for positive bases and } a \neq 0 \end{aligned}$$

This change factor is independent of x, so we have shown that x and y have constant change factors.

We have now shown both desired directions. □

Definition 4.4 (Exponential function, covariation definition)**.** A function from $\mathbb{R}$ to $\mathbb{R}$ is an exponential function if equal changes in the input result in equal ratio changes in the output.

Definition 4.5 (Exponential function, correspondence definition)**.** A function from $\mathbb{R}$ to $\mathbb{R}$ is an exponential function if it can be expressed in the form $x \mapsto ba^x$ where $a, b \in \mathbb{R}$, a is positive, and b is nonzero.

Theorem 4.6 (Two ways to define exponential functions are equivalent)**.** *The above two ways to define exponential functions are equivalent: a function satisfies the conditions of one definition if and only if it satisfies the conditions of the other definition.*

Idea of proof. By Theorem 4.3! □

Find all functions f such that they have constant change factors and $f(0) = 1$. What do the graphs of this family look like?