

## Annuities and Sinking Funds

### Sinking Fund

A sinking fund is an account earning compound interest into which you make periodic deposits. Suppose that the account has an annual interest rate of  $r$  compounded  $m$  times per year, so that  $i = r/m$  is the interest rate per compounding period. If you make a payment of  $PMT$  at the end of each period, then the future value after  $t$  years, or  $n = mt$  periods, will be

$$FV = PMT \frac{(1 + i)^n - 1}{i}.$$

### Payment Formula for a Sinking Fund

Suppose that an account has an annual rate of  $r$  compounded  $m$  times per year, so that  $i = r/m$  is the interest rate per compounding period. If you want to accumulate a total of  $FV$  in the account after  $t$  years, or  $n = mt$  periods, by making payments of  $PMT$  at the end of each period, then each payment must be

$$PMT = FV \frac{i}{(1 + i)^n - 1}.$$

### Present Value of an Annuity

An **annuity** is an account earning compound interest from which periodic withdrawals are made. Suppose that the account has an annual rate of  $r$  compounded  $m$  times per year, so that  $i = r/m$  is the interest rate per compounding period. Suppose also that the account starts with a balance of  $PV$ . If you receive a payment of  $PMT$  at the end of each compounding period, and the account is down to \$0 after  $t$  years, or  $n = mt$  periods, then

$$PV = PMT \frac{1 - (1 + i)^{-n}}{i}.$$

### Payment Formula for an Ordinary Annuity

Suppose that an account has an annual rate of  $r$  compounded  $m$  times per year, so that  $i = r/m$  is the interest rate per compounding period. Suppose also that the account starts with a balance of  $PV$ . If you want to receive a payment of  $PMT$  at the end of each compounding period, and the account is down to \$0 after  $t$  years, or  $n = mt$  periods, then

$$PMT = PV \frac{i}{1 - (1 + i)^{-n}}.$$

**Problem 1.** Suppose you deposit \$900 per month into an account that pays 4.8% interest, compounded monthly. How much money will you have after 9 months?

Solution: We want to know how much we will have in the future, so we use the formula for the future value of a sinking fund:

$$FV = PMT \frac{(1 + i)^n - 1}{i}$$

In this case  $i = \frac{0.048}{12} = 0.0004$  and  $n = 12 * \left(\frac{9}{12}\right) = 9$  (note that 9 months is  $t = \frac{9}{12}$  of a year). Thus, the future value is

$$FV = 900 * \frac{(1 + 0.0004)^9 - 1}{0.0004} = 900 * \frac{(1.0004)^9 - 1}{0.0004} = 8112.97$$

So there will be \$8,112.97 in the account after 9 months. Notice that if you just put \$900 per month into your sock drawer, you would have  $\$900 \times 9 = \$8,100$  after 9 months. The extra \$12.97 is from interest.

Calculator entry: To enter this problem into your TI calculator, you would enter it exactly as follows:  
 $900 * (1.0004^9 - 1)/0.0004$

**Problem 2.** You have a retirement account with \$2000 in it. The account earns 6.2% interest, compounded monthly, and you deposit \$50 every month for the next 20 years. How much will be in the account at the end of those 20 years?

Solution: A retirement account is a sinking fund since you are making periodic deposits. In this case, there's already \$2,000 in the account when you start making the periodic deposits. We need to treat this original \$2,000 separately – we will treat it as a separate account that earns compound interest. The formula for the future value of an account that earns compound interest is

$$FV = PV * \left(1 + \frac{r}{m}\right)^{mt}$$

For this formula,  $m$  is the number of times compounded per year (12 in this case since it's compounded monthly). So in 20 years, the \$2,000 that was already in the account will be worth

$$FV = 2000 * \left(1 + \frac{0.062}{12}\right)^{12*20} = 6889.20$$

Now the \$50 per month for 20 years is the sinking fund part, so we use the future value of a sinking fund formula to see how much that will be worth 20 years from now:

$$FV = 50 * \frac{\left(1 + \frac{0.062}{12}\right)^{12*20} - 1}{\frac{0.062}{12}} = 23657.42$$

The total amount in the account after 20 years will be the sum of what we got from the original \$2,000 and the total amount from our monthly deposits: \$6,889.20 + \$23,657.42 = \$30,546.62.

For calculator entry on the first part, it's  $2000*(1+0.062/12)^(12*20)$ .

On the second part (the sinking fund), it's  $50*((1+0.062/12)^(12*20)-1)/(0.062/12)$ . Parentheses must be exactly as I put them here.

**Problem 3.** You want to set up an education account for your child and would like to have \$75,000 after 15 years. You find an account that pays 5.6% interest, compounded semiannually, and you would like to deposit money in the account every six months. How large must each deposit be in order to reach your goal?

Solution: In this case you want to find out how much you should make in equal periodic deposits (payments) in order to have \$75,000 in the future. The unknown in this case is  $PMT$ , so we use the payment formula for a sinking fund:

$$PMT = FV \frac{i}{(1+i)^n - 1}$$

In this case, the deposits are made (and the interest is compounded) semiannually (meaning 2 times per year), so  $m = 2$ . Thus,

$$PMT = 75000 * \frac{\frac{0.056}{2}}{\left(1 + \frac{0.056}{2}\right)^{2*15} - 1} = 1628.19$$

So we must deposit \$1,628.19 every six months in order to have \$75,000 in the account after 15 years.

Calculator entry:  $75000*(0.056/2)/((1+0.056/2)^(2*15)-1)$

**Problem 3.** Tom has just won the lottery and decides to take the 20 year annuity option. The lottery commission invests his winnings in an account that pays 4.8% interest, compounded annually. Each year for those 20 years, Tom receives a check from the lottery commission for \$250,000. What is the present value of Tom's winnings? (Notice that this would be the amount that Tom would get if he chose the lump-sum option). What is the total amount of money that Tom gets over the 20 year period?

Solution: This is clearly an annuity question since it says so in the problem. We are told what the payments are for the annuity, and asked to find the present value, so we use the present value formula for an annuity:

$$PV = PMT \frac{1 - (1 + i)^{-n}}{i}$$

Since this annuity is compounded annually (and the payments are made annually),  $m = 1$  (meaning  $i = r$  and  $n = m$ ), and we get

$$PV = 250000 * \frac{1 - (1 + 0.048)^{-20}}{0.048} = 3169070.90$$

So the present value of the lottery winnings is \$3,169,070.90. Again, this is what he would get if he chose the lump-sum option. How much does he get with the annuity option? We multiply the amount he gets every year by the number of years:  $\$250,000 \times 20 = \$5,000,000$ . The difference in the lump-sum amount vs. the annuity amount is because of the interest that's earned in the annuity account. Of course, the lottery commission will advertise the annuity amount since it's greater.

Calculator entry:  $250000 * (1 - (1 + 0.048)^{-20}) / 0.048$ . Note that there's two different minus signs on the TI calculators. One is used to subtract a number from another, and the other, the one that's marked (-), is used to make a number negative (so that's the one you would use on the -20 in the exponent).

**Problem 4.** Beth has just received an inheritance of \$400,000 and would like to be able to make monthly withdrawals over the next 15 years. She decides on an annuity that pays 6.7%, compounded monthly. How much will her monthly payments be in order to draw the account down to zero at the end of 15 years?

Solution: Since Beth will be making periodic withdrawals from an account, this is an annuity question. She would like to know how much each withdrawal will be so that the entire inheritance will be gone after 15 years. We use the payment formula for an annuity to find out how much each withdrawal (payment) will be:

$$PMT = PV \frac{i}{1 - (1 + i)^{-n}} = 400000 * \frac{\frac{0.067}{12}}{1 - \left(1 + \frac{0.067}{12}\right)^{-12*15}} = 3528.56$$

Thus, each withdrawal will be \$3,528.56. At the end of the 15 years, nothing will be left.

Calculator:  $400000*(0.067/12)/(1-(1+0.067/12)^{-12*15})$

**Problem 5.** Ozzy is working in a tire factory that offers a pension in the form of an annuity that pays 5% annual interest, compounded monthly. He wants to work for 30 years and then have a retirement income of \$4000 per month for 25 years. How much do he and his employer together have to deposit per month into the pension fund to accomplish this?

Solution: This problem is probably the most realistic, and most closely matches what a typical person will do in his or her life (save money during their working life, then spend that money during retirement). The only thing we know is what Ozzy wants to have during retirement: \$4,000 per month for 25 years. Since this is money he will be withdrawing from an account, it is an annuity. We would first like to know how much money he needs in order to be able to make these monthly withdrawals for 25 years. Thus, we need the present value of an annuity:

$$PV = PMT \frac{1 - (1 + i)^{-n}}{i} = 4000 * \frac{1 - \left(1 + \frac{0.05}{12}\right)^{-12*25}}{\frac{0.05}{12}} = 684240.19$$

Thus, Ozzy will need \$684,240.19 to fund his retirement annuity. How is he going to get this money? He can do it by depositing some amount of money each month into a sinking fund for his entire 30 year working life. How much does he need to deposit into his sinking fund each month? We can use the payment formula for a sinking fund to find out. We use the present value of the annuity as the future value of the sinking fund:

$$PMT = FV \frac{i}{(1 + i)^n - 1} = 684240.19 * \frac{\frac{0.05}{12}}{\left(1 + \frac{0.05}{12}\right)^{12*30} - 1} = 822.15$$

Thus, Ozzy will need to deposit \$822.15 per month into his sinking fund (retirement account) for 30 years in order to get the \$684,240.19 that he will need to be able to fund his retirement of \$4,000 per month for 25 years. Of course, in real life not many people are able to set aside over \$800 per month, but most employers that offer a retirement plan will match your contribution, so Ozzy will probably only need to put in half that and his employer putting in the rest.

Calculator for the annuity formula:  $4000 * (1 - (1 + 0.05/12)^{-12*25}) / (0.05/12)$

For the sinking fund formula:  $684240.19 * (0.05/12) / ((1 + 0.05/12)^{12*30} - 1)$

**Problem 6.** Chris and Katie are buying a house and have taken out a 30 year, \$200,000 mortgage at 6.8% interest, compounded monthly. What will their monthly payments be? How much money will they have actually paid at the end of 30 years? How much interest will they have paid?

Solution: This problem is also quite realistic. Most people will need to borrow money from a bank (or other lender) in order to buy a house, and the loan they receive is called a mortgage. A mortgage is an annuity that a bank purchases from you, the borrower. It seems odd to look at it that way, but that's really what's happening. The bank is investing money into the house that you are buying, and what do they get in return? They get the interest that you pay on the mortgage. So from the bank's point of view, they are putting money into an account (by loaning it to you), and that account pays interest (the interest rate that you pay). The bank makes periodic withdrawals from the account (your monthly mortgage payments) until there is nothing left in the account, at which time the bank no longer has any ownership of the house and the borrower becomes the outright owner of the house. To find the monthly payments necessary, we use the payment formula for an annuity:

$$PMT = PV \frac{i}{1 - (1 + i)^{-n}} = 200000 * \frac{\frac{0.068}{12}}{1 - \left(1 + \frac{0.068}{12}\right)^{-12*30}} = 1303.85$$

Thus, their monthly payments will be \$1,303.85.

To find out how much they will have actually paid at the end of 30 years, we simply multiply the monthly payments by the total number of payments (12 payments per year for 30 years equals 360 payments):  $\$1,303.85 \times 360 = \$469,386.14$ . This means that they will have paid  $\$469,386.14 - \$200,000 = \$269,386.14$  in interest over the course of the 30 year mortgage. This is a *lot* of money to be paying in interest, but this is very typical for people who want to buy a house. Keep this in mind when you want to buy a house! Even very small changes in the interest rate will considerably alter the total interest you pay. Also, taking a mortgage for a shorter period of time will lessen the total interest paid, though it will most likely increase your monthly payments. If/when you look for a mortgage, you will probably notice that shorter loan periods have better (lower) interest rates, so if you can afford the higher payments, the 15 or 20 year mortgage is the better way to go if you want to save money on interest in the long run.