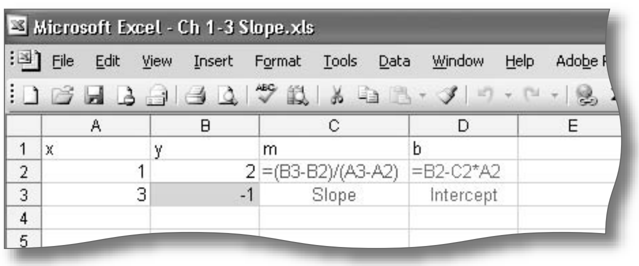
77. ● If y and x are related by the linear expression $y = mx + b$, how will y change as x changes if m is positive? negative? zero?

78. ● Your friend April tells you that $y = f(x)$ has the property that, whenever x is changed by Δx, the corresponding change in y is $\Delta y = -\Delta x$. What can you tell her about f?

79. tech Ex Consider the following worksheet:



	A	B	C	D	E
1	x	y	m	b	
2	1	2	=(B3-B2)/(A3-A2)	=B2-C2*A2	
3	3	-1	Slope	Intercept	
4					
5					

What is the effect on the slope of increasing the y-coordinate of the second point (the point whose coordinates are in Row 3)? Explain.

80. tech Ex Referring to the worksheet in Exercise 79, what is the effect on the slope of increasing the x-coordinate of the second point (the point whose coordinates are in row 3)? Explain.

● basic skills   tech Ex technology exercise

1.4 Linear Models

Using linear functions to describe or approximate relationships in the real world is called **linear modeling.** We start with some examples involving cost, revenue, and profit.

Cost, Revenue, and Profit Functions

Example 1 Linear Cost Function

As of January, 2005, Yellow Cab Chicago's rates were \$1.90 on entering the cab plus \$1.60 for each mile.*

Photodisc/Getty Images

a. Find the cost C of an x-mile trip.

b. Use your answer to calculate the cost of a 40-mile trip.

c. What is the cost of the second mile? What is the cost of the tenth mile?

d. Graph C as a function of x.

Solution

a. We are being asked to find how the cost C depends on the length x of the trip, or to find C as a function of x. Here is the cost in a few cases:

Cost of a 1-mile trip: $C = 1.60(1) + 1.90 = 3.50$ 1 mile @ \$1.60 per mile plus \$1.90

Cost of a 2-mile trip: $C = 1.60(2) + 1.90 = 5.10$ 2 miles @ \$1.60 per mile plus \$1.90

Cost of a 3-mile trip: $C = 1.60(3) + 1.90 = 6.70$ 3 miles @ \$1.60 per mile plus \$1.90

Do you see the pattern? The cost of an x-mile trip is given by the linear function:

$$C(x) = 1.60x + 1.90$$

Notice that the slope 1.60 is the incremental cost per mile. In this context we call 1.60 the **marginal cost;** the varying quantity $1.60x$ is called the **variable cost.** The

*According to their website at www.yellowcabchicago.com/.

C-intercept 1.90 is the cost to enter the cab, which we call the **fixed cost.** In general, a linear cost function has the following form:

$$C(x) = \overbrace{\underset{\substack{\uparrow\\ \text{Marginal cost}}}{m}x}^{\text{Variable cost}} + \underset{\substack{\uparrow\\ \text{Fixed cost}}}{b}$$

b. We can use the formula for the cost function to calculate the cost of a 40-mile trip as:

$$C(40) = 1.60(40) + 1.90 = \$65.90$$

c. To calculate the cost of the second mile, we *could* proceed as follows:

Find the cost of a 1-mile trip: $C(1) = 1.60(1) + 1.90 = \$3.50$

Find the cost of a 2-mile trip: $C(2) = 1.60(2) + 1.90 = \$5.10$

Therefore, the cost of the second mile is $\$5.10 - \$3.50 = \$1.60$

But notice that this is just the marginal cost. In fact, the marginal cost is the cost of each additional mile, so we could have done this more simply:

Cost of second mile = Cost of tenth mile = Marginal cost = $1.60

d. Figure 16 shows the graph of the cost function, which we can interpret as a *cost vs. miles* graph. The fixed cost is the starting height on the left, while the marginal cost is the slope of the line.

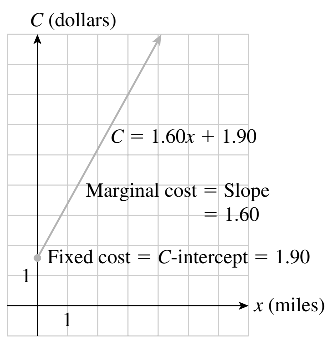


Figure 16

Before we go on... In general, the slope m measures the number of units of change in y per 1-unit change in x, so *we measure m in units of y per unit of x:*

Units of Slope = Units of y per unit of x

In Example 1, y is the cost C, measured in dollars, and x is the length of a trip, measured in miles. Hence,

Units of Slope = Units of y per Unit of x = Dollars per mile

The y-intercept b, being a value of y, is measured in the same units as y. In Example 1, b is measured in dollars. ■

Here is a summary of the terms used in the preceding example, along with an introduction to some new terms.

Cost, Revenue, and Profit Functions

A **cost function** specifies the cost C as a function of the number of items x. Thus, $C(x)$ is the cost of x items. A cost function of the form

$$C(x) = mx + b$$

is called a **linear cost function.** The quantity mx is called the **variable cost** and the intercept b is called the **fixed cost.** The slope m, the **marginal cost,** measures the incremental cost per item.

mathematics At Work

Esteban Silva

TITLE Owner
INSTITUTION Regimen


Patrick Farace

Regimen is a retail shop and online merchant of high-end men's grooming products, a small business venture under my development. I came up with this concept in order fill the growing demand for men's grooming products from both graying baby boomers wanting to retain their competitive edge and young men who are increasingly accepting of the idea that is essential to be well styled and well groomed. The currently $3.5 billion a year men's grooming market is ever-expanding and there is tremendous opportunity for Regimen to take advantage of this untapped potential.

In the initial stages of this business venture I have relied on math to calculate the amount of capital needed to launch and sustain the business until it becomes profitable. Using spreadsheets I input projected sales figures and estimated monthly expenses to formulate if it is possible to realistically meet targets and achieve break-even in a timely matter. With assistance from a professional interior designer, I have drawn up plans which include space acquisition, contracting, and construction costs in order budget for build-out expenses.

I have teamed up with Yahoo! Small Business Solutions and devised an online advertising strategy which allows me to reach out to the niche customers my company's products are geared towards. Using a sponsored search method of advertising I pre-determine how much I am willing to spend for each combination of keywords which drive traffic onto my website via Yahoo! I can track on a daily basis the number of matches each combination of keywords are receiving and, therefore, determine if any of them need to be altered. It's very important that I analyze these figures frequently so I can redirect the limited marketing resources of this start-up company into the most effective channels available. Thankfully, the applied mathematics techniques I learned in college have helped me live the dream of owning my own business and being my own boss.

The **revenue** resulting from one or more business transactions is the total payment received, sometimes called the gross proceeds. If $R(x)$ is the revenue from selling x items at a price of m each, then R is the linear function $R(x) = mx$ and the selling price m can also be called the **marginal revenue.**

The **profit,** on the other hand, is the *net* proceeds, or what remains of the revenue when costs are subtracted. If the profit depends linearly on the number of items, the slope m is called the **marginal profit.** Profit, revenue, and cost are related by the following formula:

$$\text{Profit} = \text{Revenue} - \text{Cost}$$
$$P = R - C$$

If the profit is negative, say $-\$500$, we refer to a **loss** (of \$500 in this case). To **break-even** means to make neither a profit nor a loss. Thus, break-even occurs when $P = 0$, or

$$R = C \qquad \text{Break-even}$$

The **break-even point** is the number of items x at which break-even occurs.

quick Example

If the daily cost (including operating costs) of manufacturing x T-shirts is $C(x) = 8x + 100$, and the revenue obtained by selling x T-shirts is $R(x) = 10x$, then the daily profit resulting from the manufacture and sale of x T-shirts is

$$P(x) = R(x) - C(x) = 10x - (8x + 100) = 2x - 100$$

Break-even occurs when $P(x) = 0$, or $x = 50$.

Example 2 Cost, Revenue, and Profit

The manager of the FrozenAir Refrigerator factory notices that on Monday it cost the company a total of \$25,000 to build 30 refrigerators and on Tuesday it cost \$30,000 to build 40 refrigerators.

a. Find a linear cost function based on this information. What is the daily fixed cost, and what is the marginal cost?

b. FrozenAir sells its refrigerators for \$1500 each. What is the revenue function?

c. What is the profit function? How many refrigerators must FrozenAir sell in a day in order to break even for that day? What will happen if it sells fewer refrigerators? If it sells more?

Solution

a. We are seeking C as a linear function of x, the number of refrigerators sold:

$$C = mx + b$$

We are told that $C = 25{,}000$ when $x = 30$, and this amounts to being told that (30, 25,000) is a point on the graph of the cost function. Similarly, (40, 30,000) is another point on the line (Figure 17).

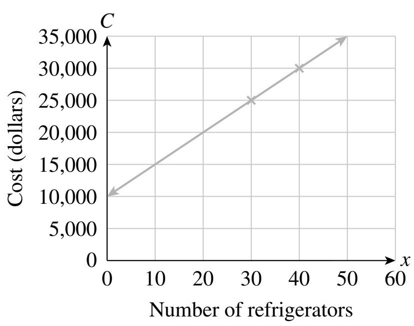


Figure 17

We can now use the point-slope formula to construct a linear cost equation. Recall that we need two items of information: a point on the line and the slope:

- ***Point*** Let's use the first point: $(x_1, C_1) = (30{,}25{,}000)$ — *C* plays the role of *y*
- ***Slope*** $m = \dfrac{C_2 - C_1}{x_2 - x_1} = \dfrac{30{,}000 - 25{,}000}{40 - 30} = 500$ — Marginal cost = \$500

The cost function is therefore

$$C(x) = 500x + b$$

where $b = C_1 - mx_1 = 25{,}000 - (500)(30) = 10{,}000$ — Fixed cost = \$10,000

so $C(x) = 500x + 10{,}000$

Because $m = 500$ and $b = 10{,}000$ the factory's fixed cost is \$10,000 each day, and its marginal cost is \$500 per refrigerator.

b. The revenue FrozenAir obtains from the sale of a single refrigerator is \$1500. So, if it sells x refrigerators, it earns a revenue of

$$R(x) = 1500x$$

c. For the profit, we use the formula

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

For the cost and revenue, we can substitute the answers from parts (a) and (b) and obtain

$$
\begin{aligned}
P(x) &= R(x) - C(x) && \text{Formula for profit}\\
&= 1500x - (500x + 10{,}000) && \text{Substitute } R(x) \text{ and } C(x)\\
&= 1000x - 10{,}000
\end{aligned}
$$

Here, $P(x)$ is the daily profit FrozenAir makes by making and selling x refrigerators. Finally, to break even means to make zero profit. So, we need to find the x such that $P(x) = 0$. All we have to do is set $P(x) = 0$ and solve for x:

$$1000x - 10{,}000 = 0$$

giving

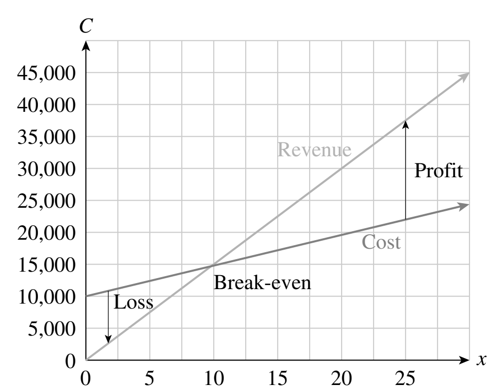
$$x = \frac{10{,}000}{1000} = 10$$

To break even, FrozenAir needs to manufacture and sell 10 refrigerators in a day.

For values of x less than the break-even point, 10, $P(x)$ is negative, so the company will have a loss. For values of x greater than the break-even point, $P(x)$ is positive, so the company will make a profit. This is the reason why we are interested in the point where $P(x) = 0$. Since $P(x) = R(x) - C(x)$, we can also look at the break-even point as the point where Revenue = Cost: $R(x) = C(x)$ (see Figure 18).

+ *Before we go on...* We can graph the cost and revenue functions from Example 2, and find the break-even point graphically (Figure 18):

$$\text{Cost: } C(x) = 500x + 10{,}000$$
$$\text{Revenue: } R(x) = 1500x$$



Break-even occurs at the point of intersection of the graphs of revenue and cost.

Figure **18**

The break-even point is the point where the revenue and cost are equal—that is, where the graphs of cost and revenue cross. Figure 18 confirms that break-even occurs when $x = 10$ refrigerators. Or, we can use the graph to find the break-even point in the first place. If we use technology, we can "zoom in" for an accurate estimate of the point of intersection.

Excel has an interesting feature called "Goal Seek" that can be used to find the point of intersection of two lines numerically (rather than graphically). The downloadable Excel tutorial for this section contains detailed instructions on using Goal Seek to find break-even points. ■

Demand and Supply Functions

The demand for a commodity usually goes down as its price goes up. It is traditional to use the letter q for the (quantity of) demand, as measured, for example, in weekly sales. Consider the following example.

Example 3 Linear Demand Function

You run a small supermarket, and must determine how much to charge for Hot'n'Spicy brand baked beans. The following chart shows weekly sales figures for Hot'n'Spicy at two different prices.

Price/Can (p)	\$0.50	\$0.75
Demand (cans sold/week) (q)	400	350

a. Model the data by expressing the demand q as a linear function of the price p.

b. How do we interpret the slope? The q-intercept?

c. How much should you charge for a can of Hot'n'Spicy beans if you want the demand to increase to 410 cans per week?

Solution

a. A **demand equation** or **demand function** expresses demand q (in this case, the number of cans of beans sold per week) as a function of the unit price p (in this case, price per can). We model the demand using the two points we are given: (0.50, 400) and (0.75, 350).

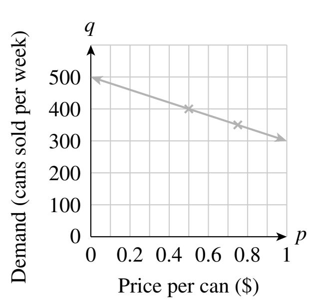


Figure 19

Point: (0.50, 400)

$$\textbf{\textit{Slope:}}\quad m = \frac{q_2 - q_1}{p_2 - p_1} = \frac{350 - 400}{0.75 - 0.50} = \frac{-50}{0.25} = -200$$

Thus, the demand equation is

$$q = mp + b$$
$$= -200p + (400 - (-200)(0.50))$$

or $\quad q = -200p + 500$

Figure 19 shows the data points and the linear model.

b. The key to interpreting the slope, $m = -200$, is to recall (see Example 1) that we measure the slope in units of y per unit of x. In this example, we mean units of q per unit of p, or the number of cans sold per dollar change in the price. Because m is negative, we see that the number of cans sold decreases as the price increases. We conclude that the weekly sales will drop by 200 cans per \$1-increase in the price.